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A Class of Convex, Polygonal Bounded Finite Elements

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Abstract

In this document different types of finite elements are presented based on polygonal bounded cells. The polygonal bounded cell is defined in the n-dimensional Euclidean space as convex cell and it's extended to a non convex formulation. For the cell a local coordinate system is introduced based on the natural coordinates. Finally the interpolation on the polygonal bounded cell is presented by using interpolation functions based on local coordinates.

1 Introduction

The method of the finite elements is a numeric method both for the interpolation of given basic values and for the numeric approximation of stationary and instationary partial differential equations. The theory is based on the formulation of suitable finite elements and element decomposition. Often used finite elements are based on triangles or squares in two-dimensional space and tetra- or hexahedron in the three-dimensional space.

The different types of finite elements are generalizable regarding their geometry and dimension. On this basis it is possible to design an element-decomposition, whose elements are independent on a fixed number of vertex or dimension. In the following, the polygonal bounded, n-dimensional cell in the Euclidean vector space are defined and extended by a non-convex formulation. For the complete description of a polygonal bounded cell a local coordinate system is introduced in respect to the points describing the cell. With the help of the local coordinate system, the interpolation over the element geometry is made possible.

2 Cell

The geometrical basis of a finite Element is formulated as a convex cell. In the context of this article, its' concept is based on the n-dimensional Euclidean vector space E^n. In the n-dimensional Euclidean vector space, the definition of a convex cell can take place in different equivalent ways [5]. Normally the convex hull is used to define a convex cell.

Let P be a non empty set of elements of a n-dimensional Euclidean vector space. The elements of the set P are called reference points. The convex cell Z is the convex hull of the reference points. The convex hull can be interpreted with the intersection of half spaces respectively to the given set of reference points. A half space is a set of elements of E^n, whose coordinates satisfy the inequation $\sum_{i=1}^{n} \lambda_i x_i \leq \gamma$ with $(\lambda_i, \ldots, \lambda_n) \neq (0, \ldots, 0)$.

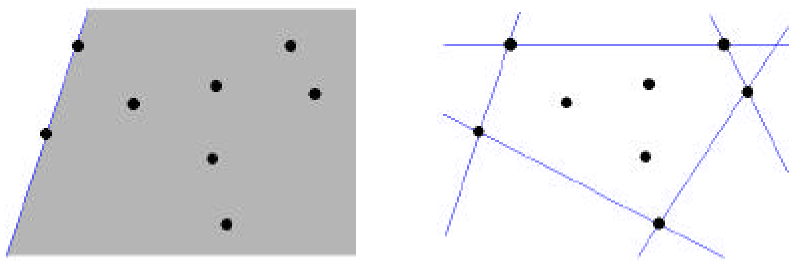

Figure 1: Half space and intersection of half spaces

The half spaces must satisfy two conditions. Firstly, the elements of the set P must be inside the half space. Secondly, there must be at least n points in the hyper plane of the n-dimensional half space. The hyper plane is described with the equation $\sum_{i=1}^{n} \lambda_i x_i = \gamma$. The intersection of half spaces is convex and describes the convex hull of the reference points P.

In each hyper plane of the half space, which is made up of convex hulls, lays a (n-1)-dimensional face of a convex cell Z. A convex cell has at the most finite faces. Each face is a convex cell. An m-dimensional convex cell $(m \leq n)$ has it's faces in the following dimension j=0,1,...,m-1. If Z is an m-dimensional convex cell, the 0-dimensional faces are called vertexes. The set of vertexes E is a subset of the given set of points P, $E \subseteq P$.

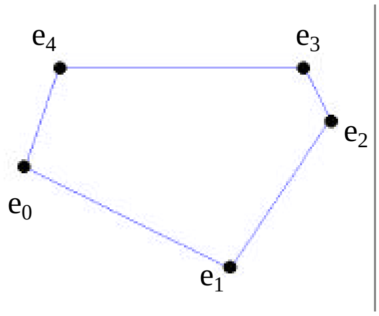

Figure 2: Convex cell

Based on the cell definition, the geometrical basis of a finite element is limited to the polygonal bounded convex cell. It's however expandable to polygonal bounded non-convex cells. Non-convex cells can be constructed from convex cells.

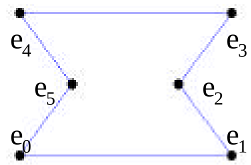

Figure 3: Non-convex cell

The construction of non-convex cells is based on the use of regularized set operations $\cup^*$, $\cap^*$, $\setminus^*$ [4] in respect of convex cells. Let the set P be the set of reference points of a non-convex cell Z. The non-convex cell Z is then constructed from the convex cell cZ of the set P and convex cells S_i. The convex cells S_i of the set S describe subsets of the cell cZ, which are not subsets of the non-convex cell Z. The non-convex cell Z is formed from the regularized difference of the cell kZ and the cells S.

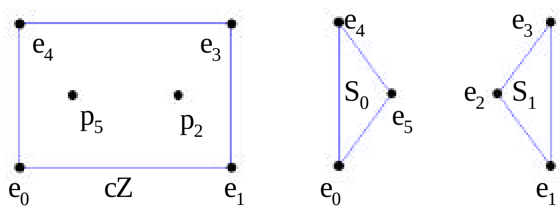

Figure 4: Convex hull and convex parts

For the use of cells as geometrical basis of finite elements it is appropriate to represent the points of a cell by means of local coordinate. A local coordinate system must be aplliable both to convex and to non-convex cells.

3 Local coordinates

The description of the points of a cell as subset of the Euclidean space $Z \subseteq E^n$ is made appropriately by a local coordinate system. The variable number of vertexes of a cell requires the formulation of a coordinate system in respect of the vertexes. A possible coordinate system of a convex cell is based on homogeneous coordinates, which result from the Voronoi-decomposition of the vertexes of a cell.

3.1 Voronoi-decomposition

A local discretisation is given through a set of reference points P. The Voronoi-decomposition is the decomposition of Euclidean space E^n relatively to this local discretisation on the basis of the neighbourhood relationship. The neighbourhood relationship is defined over the Euclidean distance d between two points $x \in E^n$ and a point of reference $a \in P$.

$$\text{x is a neighbour of a} :\Leftrightarrow d(x,y) \leq d(x,a) \forall a \in P \tag{1}$$

A point is therefore neighbour of a point of reference, if its distance to the point of reference is less than its distance to every other point of reference. The set of all points $x \in E^n$, which are neighbours of the point of reference $a \in P$, is called a Voronoi region of a and identified as R(a).

$$R(a) := \{x \in E^n \mid d(x,a) \leq d(x,p) \forall p \in P\} \tag{2}$$

The boundary of the Voronoi regions R(a) and R(b) is the set of all points, which are at the same time neighbours of the points of reference a and b. It is identified with B(a, b) and is the average of the two Voronoi regions. A point $x \in E^n$, which is the neighbour to more than n points of reference, is called Voronoi vertex. A Voronoi-vertex is the centre of the circumcircle, on its edge lies the forming points of reference of the Voronoi vertex. No further point of reference of the set P lies in the circumcircle.
Two Voronoi regions R(a) and R(b) are neighbouring regions, if they possess a nonempty boundary B(a, b). Two points of reference are neighbouring points of reference, if their Voronoi regions are neighbouring. The union of all regions is the Euclidean space. The set of regions is referred to as the Voronoi-decomposition of the Euclidean space in respect to the points of reference. The Voronoi-decomposition consists of restricted und unrestricted regions.

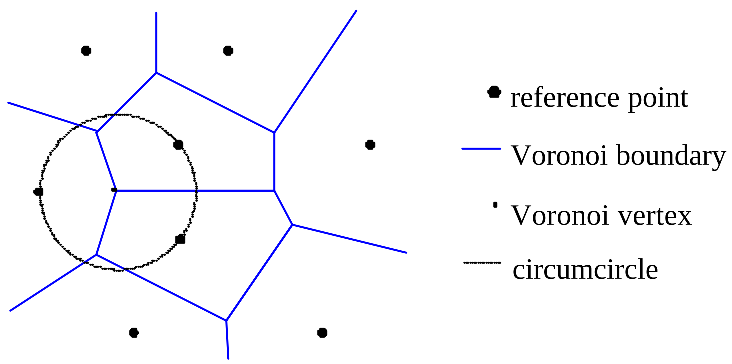

Figure 5: Voronoi-Diagram

Algorithms for the determination of the Voronoi-decomposition of a set of points of reference are for example in [1] or [8] specified. The algorithms create the Voronoi-decomposition by gradual inserting points of reference.

3.2 Natural coordinates

A point is inserted in an existing Voronoi-decomposition. Sub regions in respect of neighbouring points of reference result due the existing and the again resulting neighbourhood relations. From the proportion of the sub regions, one can determine homogeneous coordinates for the inserted point regarding its neighbouring points of reference.
Inserting a point leads to structural changes in the existing Voronoi-decomposition. The region R(x) consists of sub regions R(x, p) of neighbouring points of reference. A sub region R(x,p) contains each point of the region R(p), which is a neighbour of x, resulted from the Voronoi-decomposition of the points of reference and the point x.

$$R(x,p) := \{z \in R(p) \mid d(z,x) \le d(z,p)\} \tag{3}$$

Each sub region is assigned to a measure $\mu(R(x,p))$. This corresponds two-dimensionally to the area $\mu(R(x,p)) = A(x,p)$, and three-dimensionally to the volume $\mu(R(x,p)) = V(x,p)$ respectively.

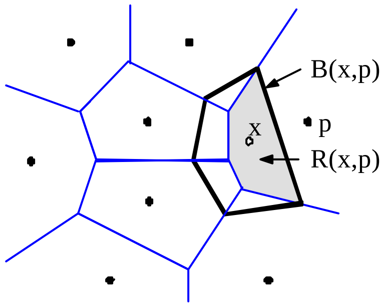

Figure 6: Sub region of the inserted point x

The relationship of the sub region R(x,p) to the region R(x) is used to describe the point x in regards to the neighbouring point of reference p.

$$\lambda(x,p) := \frac{\mu(R(x,p))}{\mu(R(x))} = \frac{\mu(R(x,p))}{\sum_{q \in P} \mu(R(x,q))} \tag{4}$$

The relationship is assignable for all neighbouring points of reference $q \in P$ of the point x. It corresponds to homogeneous coordinates of the point x in respect of its neighbours.

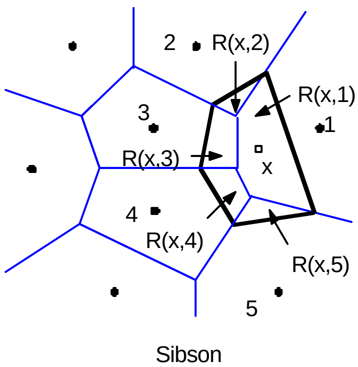

Figure 7: Construction principle of a natural coordinate

The above described coordinates of the point x are called natural coordinates [2] or Sibson-coordinates [6].
The natural coordinates take a value of one at the assigned point of reference and at all other points of reference a value of zero.

$$\lambda_i(x) = \begin{cases} 0; p \neq p_i \\ 1; p = p_i \end{cases} \tag{5}$$

The representation of the point x through the natural coordinates satisfy the local coordinate property [6].

$$x = \sum \lambda_i(x) p_i \tag{6}$$

The natural coordinates satisfy the partition of unity.

$$\sum \lambda_i = 1 \tag{7}$$

With the help of the natural coordinates it is possible to represent each points of the convex hull of the given point set P in dependence of its neighbours.

3.3 Convex cell

Due to the characteristics of the natural coordinates, the points of a convex cell can de described in dependence of the vertexes of the cell. The vertexes of the convex cell and the natural coordinates which are formed on the basis of the vertexes, are called local coordinate system of the convex cell.
For the determination of the local coordinates of any point p within the cell its natural coordinates are determined concerning the vertexes of the cell. Inserting the point x in the Voronoi-decomposition of the vertexes, creates the sub regions R(x,e), $e \in E$. The measure of the sub region of a vertex e in relation to the measure of the region R(x) determines the influence of a vertex e on the point x. A point x of the convex cell possesses a local coordinate λ concerning to each vertex of the convex cell.

$$\lambda_i(x, e_i) = \frac{\mu(R(x, e_i))}{\mu(R(x))} \tag{8}$$

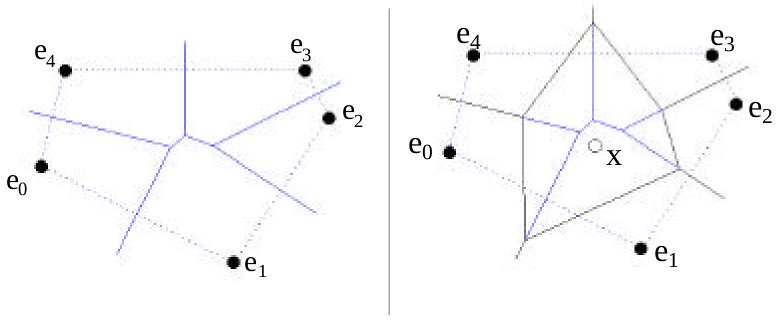

Figure 8: Voronoi-decomposition of convex cell and sub regions of vertexes with respect to x

If the point lies on a face or outside of the convex cell the determination of the local coordinate separately must be regarded. If a point x lies outside, its natural and consequently local coordinates are not defined because this results to an infinite value of the sub regions in respect of the vertexes. If the point x lies on a face of the convex cell, this results to an infinite value of the vertexes of that face. The influence of the vertexes can be determined by calculating the limit values [7]. The point x is then only affected by the vertexes of the side. The local coordinates of the remaining vertexes have the zero value.

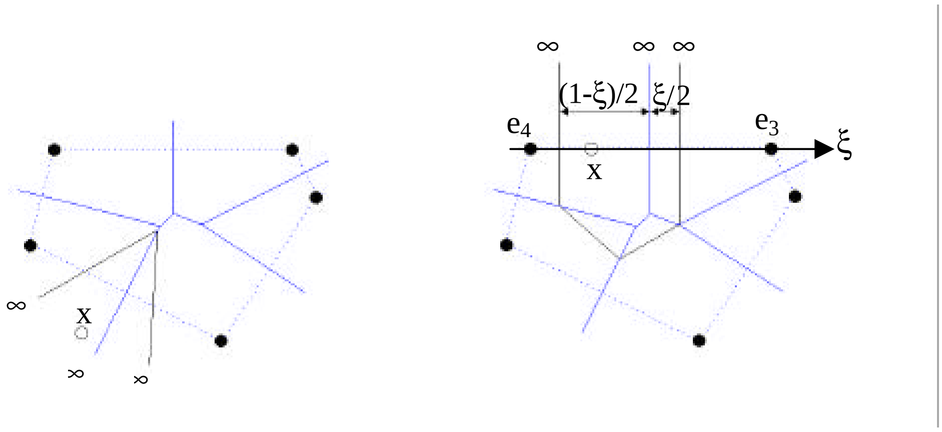

Figure 9 : Point x out of cell and on a face of cell

Due to the construction of the local coordinates of a convex cell by means of the neighbours, the local coordinates of a point x does not depend inevitably on each vertex of the cell. If the point x does not possess a vertex as a neighbour, no sub region develops for that point and that vertex. The coordinate of the point has no value with respect to the vertex. The sphere of influence of a vertex in the cell is determined by the Voronoi-circumcircles, on which the vertex is involved. Therefore the neighbourhoods of the vertexes play a substantial role in the construction of the local coordinates of a point x.

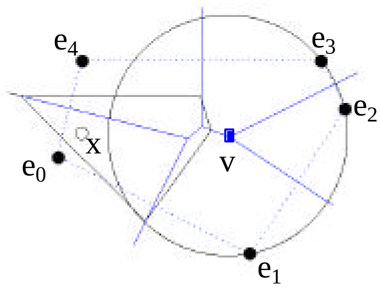

Figure 10: Edge without influence on x

3.4 Non convex cell

The local coordinate system of a non-convex cell have to show the same characteristics for the general use of the cell as those of a convex cell. The local coordinate system of convex cells is transferable to a non-convex cell only by an adjustment of the construction. The local coordinate system of the convex cell is based on the Voronoi-decomposition of the vertexes. The neighbours between the vertexes, resulting from the Voronoi-decomposition, consider the space of the convex hull with respect to the vertexes. When the coordinates construction of the convex cell are directly transferred to that of non-convex cells, too many neighbours are considered.

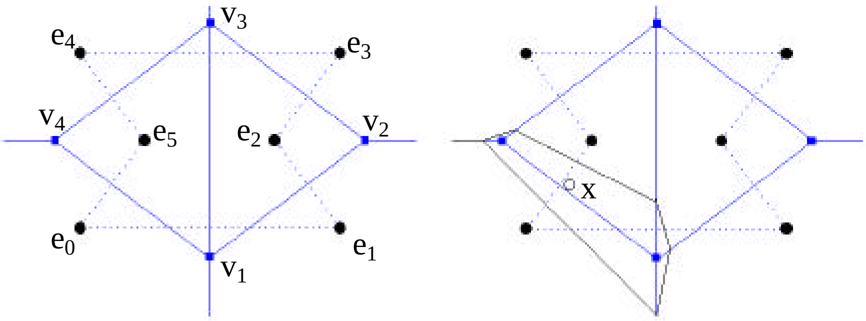

Figure 11: Voronoi-Decomposition of a non convex cell

For example, in the non-convex cell of the figure 11 are the vertexes e_0 and e_4 neighbousr due to the Voronoi-decomposition. The point x which can be represented in local coordinates falls into the direct sphere of influence of both vertexes. This sphere of influence is given by the Voronoi-circumcircle, in whose construction both corners are involved. The center of this circumcircle is the Voronoi-vertex v_4. The neighbours relationship between the vertexes e_0 and e_4 , which is represented by the Voronoi-vertex v_4 , must not be considered in the non-convex cell.
When constructing the Voronoi-decomposition of the non-convex cell, only neighbours are considered, which are element of both convex and non-convex cell.

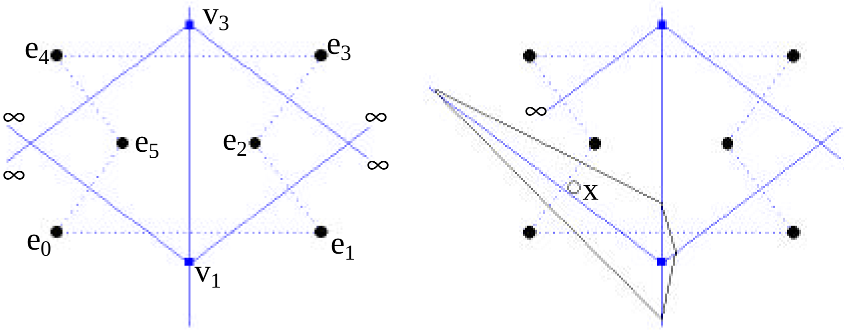

Figure 12: "Non-convex" Voronoi-decomposition of a non-convex cell

For the non-convex cell in Figure 12, the neighbours between the vertexes e_0 and e_4 as well as the vertexes e_1 and e_3 are part of the convex cell but not of the non-convex cell. These neighbours are represented by the Voronoi-vertexes v_1 and v_4 (Figure 11). If the Voronoi-vertexes are not considered, the "non-convex" Voronoi-decomposition of the vertexes are arised (Figure 12).
The bcal coordinates which resulted on the basis from the "non-convex" Voronoi-decomposition, possess the same characteristics as that of a convex cell. The general use of a cell as convex or non-convex cell is thus possible.

4 Interpolation

The definition of the cell in the Euclidean space and the formulation of the local coordinate system of a cell is the fundament for the solution of an interpolation on the cell. Consider a set of vertexes E of a cell. Each vertex is assigned to the value of the function $u(e_i)$. The values of the function u are unknown except at the vertexes.

We want to compute the function value $u(\lambda(x))$ of a given point x of the cell Z depending on the values of the vertexes of the cell. The interpolation has to determine the values at the vertexes accurately. The function value is computed by a linear combination of the values at the vertexes $e_i \in E$.

$$u(\lambda(x)) = \sum_{i=1}^{N} \phi_i(\lambda(x))u(e_i) \tag{9}$$

The linear combination is called interpolation, if the interpolation function of a vertex e_i is zero at all other vertexes and one at the vertex e_i.
Local coordinates are used as interpolation function ϕ_i for the solution of the interpolation (9).

$$\phi_i(\lambda_1,\ldots,\lambda_n) = \lambda_i(x) \tag{10}$$

The interpolation functions are C^0 at the vertexes [3] and at least C^1 within the cell. On the side of an n-dimensional cell, the interpolation corresponds to the interpolation of the (n-1)-dimensional cell. In a one-dimensional cell, the boundary is linear interpolated. If the cell is a triangle, the interpolation is equals to the linear interpolation. If the cell is a square, the interpolation is equals to the bilinear interpolation.
The construction of interpolation functions of higher order is possible on the basis of the local coordinate system.

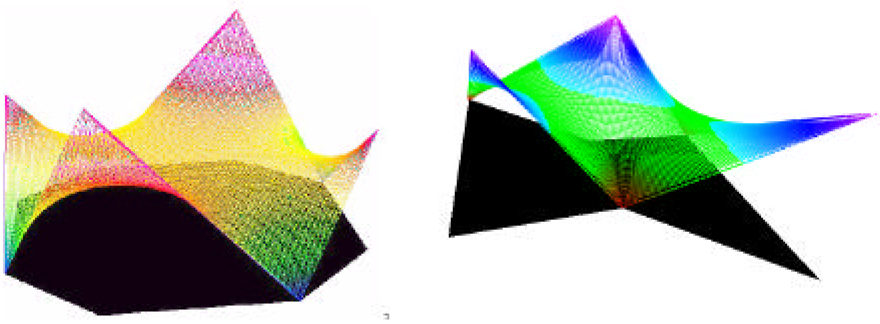

Figure x: Interpolation in a convex cell an a non-convex cell

5 Conclusion

The polygonal bounded cell serves as geometrical basis for a generalized view of different types of finite elements. The cell can be both convex and non-convex bounded. The local coordinate system of the cell makes the interpolation on the cell possible. Lokal coordinates are used for the interpolation functions of the interpolation. The formulation of interpolation functions of higher order is a subject of current research.
The presented views in connection with interpolation of higher order make it possible to formulate parametric cells based on local coordinates. The formulation of finite elements on the basis of local coordinates expects their use in a multiplicity of application in the future.

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