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## Short Communication

# Stabilized finite element method for simulation of freeway traffic

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The modeling of traffic flow is a key tool to simulate and predict the behavior of traffic systems. Macroscopic traffic simulation models are based on advection dominated coupled non-linear partial differential equations. The solution of such advection dominated equations with the method of finite elements is leading to the development of stabilization techniques. The choice of suitable stabilization parameters is often application-dependent. A stabilized finite element procedure on the basis of a Galerkin/least-square approximation is presented for systems of transient advection-dominated equations. A general rule for computing suitable element stabilization parameters is outlined which uses the spectral radius of the differential operators and the specific element expansion. The application of this approximation to a macroscopic traffic model shows the applicability of this approach. Simulation results of typical phenomena of jam formation in freeway traffic are presented.

*Keywords*: Stabilized finite elements; Macroscopic traffic model; Galerkin approximation; Super method

## 1. Introduction

In the recent years, detector equipments and variable message sign systems have been installed to control and influence traffic flows on freeways. Traffic control systems are based on the idea to avoid traffic instabilities and to homogenize the traffic flow in such a way that the risk of accidents is minimized and the mean velocity or the traffic flow is maximized. The traffic control systems need the evaluation of the measured traffic data, a short time prediction of the traffic situation and traffic simulations for possible control measures without significant time delay. Macroscopic traffic simulation models based on of Navier–Stokes-like equations (Lighthill and Whitham 1955). The numerical approximation of such advection dominated problems with the method of finite differences, finite volumes or the method of finite elements frequently leads to instabilities or to reduced accuracy of the approximated solution.

The stabilized finite element method adds mesh dependent terms to the usual Galerkin method to overcome most of the limitations in the Galerkin method by solving transport dominant problems. Since Brook and Hughes (1982) suggested stabilization with the SUPG method, different stabilizing procedures were developed for the scalar transport equation and extended to multi-scale problems. Stabilized finite element methods have grown popular over the last years, especially in application to fluid dynamics. In the work presented here, the approximation is based on stabilization with a combination of the Galerkin and least-squares approach (Christie *et al.* 1976). The choice of suitable stabilization parameters is difficult and often application-dependent (Hughes *et al.* 1989).

## 2. Statement of the problem

The following general transient problem shall be viewed. Let $\Omega$ represent the open bounded domain in $\mathbb{R}^n$ and $\Gamma$ its boundary. Find a vector valued function $U : \Omega \longrightarrow \mathbb{R}^m$ such that

$$\frac{\partial U}{\partial t} + LU + S = 0 \tag{1}$$

is valid, where $L$ is a quasi-linear differential operator and $S$ are source and sink terms. We assume that all necessary

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boundary and initial conditions which guarantee the existence of the solution are available.

Now we assume that the quasi-linear differential operator $L$ can be understood as sum of an advection operator $L_{\mathrm{adv}}$ and a diffusion operator $L_{\mathrm{diff}}$

$$L = L_{\mathrm{adv}} + L_{\mathrm{diff}} \tag{2}$$

with the following concrete form:

$$L = A_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial}{\partial x_j}\right) \tag{3}$$

here $A_i$ is the $i$ th Euler Jacobian matrix and $K_{ij}$ is the diffusivity matrix. Each operator can be divided again into its local components. As this can be shown for the transport operator:

$$L_{\mathrm{adv}} = \sum L_{\mathrm{adv},i} = \sum A_i \frac{\partial}{\partial x_i} \tag{4}$$

## 3. Semi-discrete stabilized finite element approximation

In order to approximate the equation (1) with the finite element method the domain $\Omega$ is discretized into $n_{\mathrm{el}}$ finite elements $\Omega_{\mathrm{e}}$. Let $H^1(\Omega)$ the usual Sobolev space of functions with square-integrable values and derivatives on $\Omega$.

The derivation of the semi-discrete stabilized finite element approximation is carried out via the combination of a standard Galerkin approximation and the least squares approximation. This can be described roughly, for the differential equation (1) as follows:

$$\int_{\Omega}(U_t + LU + S)\cdot w\,\mathrm{d}\Omega + \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e}(L\cdot w)(U_t + LU + S)\mathrm{d}\Omega^e = 0 \tag{5}$$

The first integral contains the Galerkin approximation (interior and boundary) and the second term contains the least-squares stabilization which is composed of the sum of integrals over the element interiors. This approximation is called semi-discrete Galerkin/least-squares method. We use the following modified semi-discrete streamline upwind Petrov–Galerkin method which is a predecessor to the Galerkin/least-squares method:

$$\int_{\Omega}(U_{,t} + LU + S)\cdot w\,\mathrm{d}\Omega + \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e}(L_{\mathrm{adv}}\cdot w)(U^G_{,t} + LU + S)\mathrm{d}\Omega^e = 0 \tag{6}$$

where $U^G_{,t}$ is determined by the standard Galerkin-method. The difference to the Galerkin/least-squares is that rather than having $L$ operating on the weighting space, only its advective part, $L_{\mathrm{adv}}$, acts there. The element stabilization parameter $\tau_e$ weighted the portion of the least-square part to the Galerkin part of the method.

## 4. Stabilization parameter

The element stabilization parameter $\tau_{\mathrm{e}}$ plays an important role for the stability and consistency as well as for the accuracy of the approximation. The derivation of the stabilization parameter is reasonably clear in the case of a steady-state one-dimensional scalar valued problem. For following advection diffusion equation in (0,1)

$$v\cdot c_x - \varepsilon\cdot c_{xx} = 1 \quad \text{with} \quad c(0) = c(1) = 0 \tag{7}$$

the element parameter $\tau_{\mathrm{e}}$ is chosen on the basis of finite difference considerations (Christie *et al.* 1976) and has the form:

$$\tau_{\mathrm{e}} := \alpha_{\mathrm{opt}}\frac{1}{2}\frac{\Delta x}{|v|} \tag{8}$$

where $\Delta x$ is the length of the domain discretization, $|v|$ the absolute value of the transport velocity. The optimality parameter $\alpha_{\mathrm{opt}}$ is computed by:

$$\alpha_{\mathrm{opt}} := \coth(Pe) - \frac{1}{Pe} \tag{9}$$

based on the Peclet number

$$Pe := \frac{|v|\cdot\Delta x}{|\varepsilon|}. \tag{10}$$

The transfer to the scalar valued multidimensional case takes place in analogy via the consideration along the characteristics, using the Euclidean norm $\|\vec{v}\|$ of the velocity vector and a characteristic element expansion $h_{\mathrm{e}}$ associated with this vector (see figure 1).

$$\tau_{\mathrm{e}} := \alpha_{\mathrm{opt}}\frac{1}{2}\frac{h_{\mathrm{e}}}{\|\vec{v}\|} \tag{11}$$

The computation of the element expansion $h_{\mathrm{e}}$ presuppose that in a particular element the velocity components $v_{\mathrm{x}}$ and $v_{\mathrm{y}}$ are substantially constant.

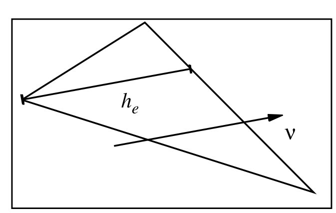


Figure 1. Computation of the element size.

### 4.1 Multi-dimensional vector valued problem

Consequently for multidimensional vector valued transport problems a norm $\|\cdot\|$ of the transport operator and a related characteristic element expansion $h_e$ are used:

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\|L_{adv}\|} \tag{12}$$

The optimality parameter $\alpha_{opt}$ is computed in the same way as in equation (9)

$$\alpha_{opt} := \coth(Pe) - \frac{1}{Pe}$$

but now the element Peclet number depends on the operator norms of the advection and diffusion differential operator

$$Pe := \frac{\|L_{adv}\| h_e}{\|L_{diff}\|}. \tag{13}$$

The choice of a suitable operator norm has a large influence on the quality of the solution. On the basis of the general definition (Kolmogorov and Fomin 1975) of the norm of continuity operators in (Euclidean) normed spaces we define the following operator norm. The differential operator has the form presented in equation (4)

$$L_{adv} = \sum L_{adv,i} = \sum A_i \frac{\partial}{\partial x_i}.$$

This leads to the operator norm

$$\|L_{adv}\| := \sqrt{\sum \rho(A_i)^2} \tag{14}$$

where $\rho(A_i)$ is the spectral radius of the operator component

$$\rho(A_i) := \max|\lambda_j(A_i)|. \tag{15}$$

Here $\lambda_{max}(A_i)$ is the absolutely largest eigen value of the Matrix $A_i$. This definition is consistent in all dimensions, starting by the one-dimensional scalar valued advective diffusive problem up to more dimensional and vector valued problems.

This approach for the stabilization parameter leads to very good numerical results for a large number of simulation models for hydro- and morphodynamic processes (Milbradt 2002) as well as macroscopic freeway traffic flows as presented in the following.

## 5. Macroscopic traffic flow modeling

Traffic flow models are essential tools to assess and control traffic flows on main freeways. The idea to model the traffic flow in a macroscopic way based on Lighthill and Whitham (1955). In this case the traffic flow is a continuous flow where individual vehicles can be identified. The continuous traffic flow can be described by the mean velocity $V$ and the traffic density $\rho$. The equations of the traffic flow model by Kühne *et al*. (1996) are very similar to the Navier–Stokes equations.

$$\frac{\partial \rho}{\partial t} = -V\frac{\partial \rho}{\partial x} - \rho\frac{\partial V}{\partial x}$$
$$\frac{\partial V}{\partial t} = -V\frac{\partial V}{\partial x} - \frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 V}{\partial x^2} + \frac{1}{\tau}(V_e - V) \tag{16}$$

In contrast to fluid dynamics, in traffic dynamics the law of conservation of momentum is not valid. The term $(V_e - V)/\tau$ is called the adaptation term or relaxation term. It is assumed that the current velocity *V(x, t)* is adapted to a prescribed equilibrium velocity $V_e$ within a certain time $\tau$. The equilibrium velocity $V_e$, depending at least on the density and the adaptation time are the most important parameters of all macroscopic traffic models. The model is characterized by a density dependent pressure parameterisation with constant velocity of propagation $c_0^2$ and due to an additional diffusion term, with constant viscosity $\mu$ reflecting some observations of viscous traffic flow in reality.

The above one-dimensional vector valued advection diffusion problem (16) can be expressed as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} - B\frac{\partial^2 U}{\partial x^2} - S = 0 \tag{17}$$

with

$$U = \begin{bmatrix} \rho \\ V \end{bmatrix}, \quad A = \begin{bmatrix} V & \rho \\ \frac{c_0^2}{\rho} & V \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\mu}{\rho} \end{bmatrix} \quad \text{and}$$

$$S = \begin{bmatrix} 0 \\ \frac{V_e - V}{\tau} \end{bmatrix}.$$

Now we have the following operator-norms

$$\left\| A\frac{\partial}{\partial x} \right\| = |V| + |c_0| \quad \text{and} \quad \left\| B\frac{\partial^2}{\partial x^2} \right\| = \frac{\mu}{\rho}$$

as well as the element stabilization parameter

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{\Delta x}{|V| + |c_0|}. \tag{18}$$

Here is $\alpha_{opt} := \coth(Pe) - 1/Pe$, and the element Peclet number can be determined by

$$Pe := \frac{(|V| + |c_0|)\Delta \cdot x}{\mu/\rho}. \tag{19}$$

The complexity of the arising phenomena is demonstrated in an academic case example. A closed single lane is considered. It is assumed that the initial traffic state is in equilibrium state with a prescribed density $\rho_e$ and a small local perturbation of the uniform density distribution. The properties of the jam formations are studied in detail depending on the density $\rho_e$.

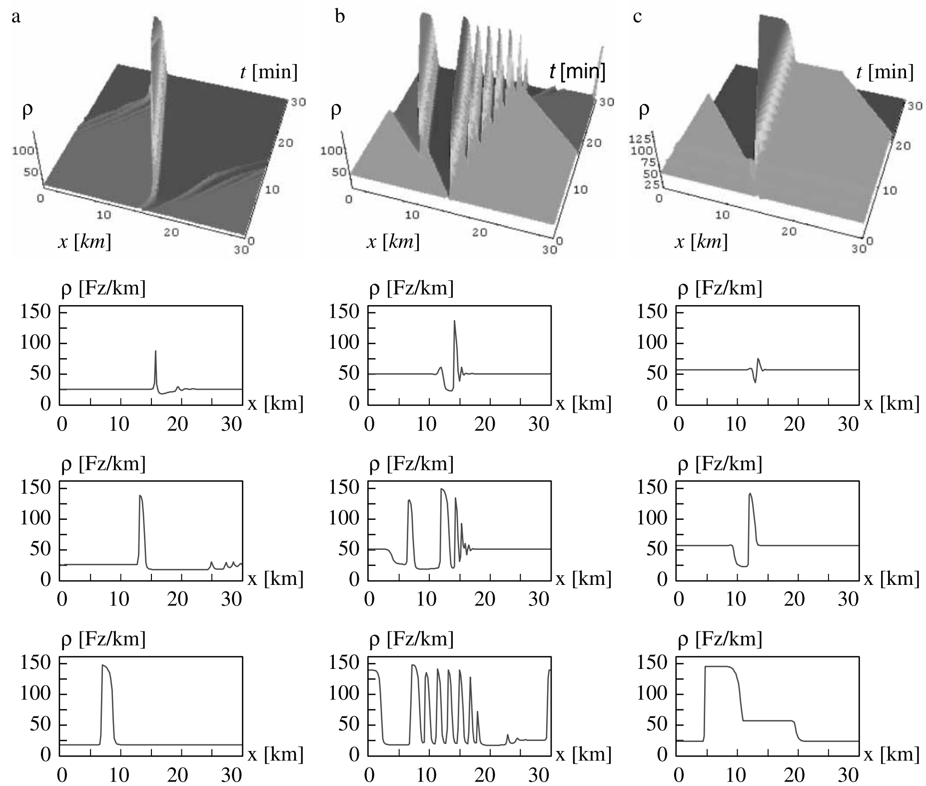


Figure 2. Density evolutions with $\rho_e = 25$ veh/km: (a), $\rho_e = 50$ veh/km; (b) $\rho_e = 56.25$ veh/km; and (c) in space-time diagram and spatial density distributions at time $t = 3$, 11, 30 min.

Figure 2 shows some typical results of a numerical perturbation analysis for macroscopic traffic flow models. The equilibrium is stable for free traffic flow with $\rho_e < 15$ veh/km and for dense traffic flow with $\rho_e > 60$ veh/km. In the stable case the perturbations of the equilibrium state decrease with increasing time. The equilibrium is unstable for congested traffic flow. There are three different phenomena of congested traffic flow with respect to jam formations: moving jams, stop and go waves and wide jams. The jams move backward with a velocity of about $-15 \pm 5$ km/h. The different phenomena of jam formation can also be observed in real traffic on highways.

The numerical realization of the Navier–Stokes-like equations with the stabilized finite element method leads to the same results as a nearly exact implicit finite difference method used by Kerner *et al.* (1996).

Practical applications include data evaluation and data editing of data obtained from local traffic measurements on the german freeway A5 near Frankfurt and traffic flow simulation for three selected traffic scenarios are implemented. It can be shown that the traffic flow model is able to reflect these typical traffic scenarios like a fixed traffic jam or a moving traffic jam (Rose 2004).


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