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A stabilized finite element method for systems of instationary advection-dominated equations in multi-dimensional domains

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Abstract
The solution of advection-dominated equations with the method of finite elements led to the development of stabilization techniques. The choice of suitable stabilization parameters is often application-dependent and difficult. A stabilized finite element procedure on the basis of a Galerkin/least-squares approximation is presented for systems of instationary advection-dominated equations in multidimensional domains. A general rule for computing suitable element stabilization parameters is outlined which uses the operator norms of the differential operators and the element expansion. The application of this approximation to solve a macroscopic traffic model and a morphodynamic model shows the applicability of this approach.

1 Introduction

Many phenomena in physics and technology relate to transport phenomena and the reaction of states and substances. Such physical and technical questions can often be represented with stationary or instationary partial differential equations. The approximation of transport dominant problems with the method of finite differences or the method of finite elements frequently leads to instabilities of the approximated solution.
To overcome most of the limitations in the Galerkin method by solving transport dominant problems , the stabilized finite element method adds mesh dependent terms to the usual Galerkin method. Stabilized finite element methods have grown popular over the last years. In the work presented here, the approximation is based on stabilization with a combination of the Galerkin and least-squares approach [1]. The choice of suitable stabilization parameters is difficult and often application-dependent.

2 Stabilized finite element approximation

The following general instationary problem shall be viewed. Let Ω represent the open bounded domain in $\mathbb{R}^n$ and Γ its boundary. Find a vector valued function $U:\Omega \rightarrow \mathbb{R}^m$ such that

$$\frac{\partial U}{\partial t} + L\,U + S = 0 \tag{1}$$

is valid, where L is a quasi-linear differential operator and S are source and sink terms. We assume that all necessary boundary and initial conditions which guarantee the existence of the solution are available.
The quasi-linear operator has the following form:

$$L \equiv A_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial}{\partial x_j}\right) \tag{2}$$

Here A_i is the i^{th} Euler Jacobian matrix and K_{ij} is the diffusivity matrix. Therefore, the operator L can be understood as sum of an advection operator L_{adv} and a diffusion operator L_{diff}:

$$L = L_{adv} + L_{diff} \tag{3}$$

each operator can be divided again into its local components with the following representation. As this can be shown for the transport operator:

$$L = \sum L_i = \sum A_i \frac{\partial}{\partial x_i} \tag{4}$$

In order to approximate the equation (1) with the finite element method the domain Ω is discretized into n_{el} finite elements Ω_e.

2.1 Semi-discrete SUPG

The derivation of the semi-discrete stabilized finite element approximation is carried out via the combination of a standard Galerkin approximation and the least squares approximation. This can be described roughly, for the differential equation (1) as follows:

$$\int_{\Omega} \left(U_{,t} + L\,U + S\right) \cdot w \; d\Omega + \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e} \left(L \cdot w\right)\left(U_{,t} + L\,U + S\right) d\Omega^e = 0 \tag{5}$$

The first integral contains the Galerkin approximation and the second term contains the least-squares stabilization which is composed of the sum of integrals over the element interiors. This approximation is called semi-discrete GLS method. We use the following modified semi-discrete SUPG method, which is a predecessor to the GLS method.

$$\int_{\Omega} \left(U_{,t} + L\,U + S \right) \cdot w \; d\Omega + \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e} \left(L_{adv} \cdot w \right) \left(U_{,t}^G + L\,U + S \right) d\Omega^e = 0 \tag{6}$$

where $U_{,t}^G$ is determined by the standard Galerkin-method. The difference to the GLS is that rather than having L operating on the weighting space, only its advective part, L_{adv}, acts there.
Obviously, an implementation in form of an predicator corrector procedure is to be realized. In the predicator step the time derivatives of the unknown variables are computed over a standard Galerkin approximation. In the corrector step the mean residue of the Galerkin approximation in the element is then computed and used by applying the transportation operator and stabilization parameter for the correction of the computed time derivative. The in such a way determined time derivatives can then be integrated over time by universal time integration procedures (Euler, Heun, Runge-Kutta, etc.). The element stabilization parameter τ_e plays an important role for the stability and consistency of the approximation.

2.2 Stabilization parameter

One-dimensional scalar valued problem

The following scalar valued one-dimensional stationary advection diffusion equation for *(0,1)*

$$v \cdot c_{,x} - \epsilon \cdot c_{,xx} = 1 \quad \text{with} \quad c(0) = c(1) = 0 \ . \tag{7}$$

is examined. In this case the element parameter τ_e is chosen on the basis of finite difference considerations and has the form:

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{\Delta x}{|v|} \tag{8}$$

where Δx is the length of the domain discretisation, v the transport velocity and α_{opt} an optimality parameter computed by

$$\alpha_{opt} := \coth(Pe) - \frac{1}{Pe} \tag{9}$$

based on the Peclet number

$$Pe := \frac{|v| \cdot h_e}{|\epsilon|} \tag{10}$$

Multi-dimensional scalar valued problem

Application in multidimensional scalar transport problems using the Euclidean norm $\|\vec{v}\|$ of the velocity vector and an element expansion h_e associated with this vector.

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\|\vec{v}\|} \tag{11}$$

Multi-dimensional vector valued problem

Consequently for multidimensional vector valued transport problems a norm $\|L_{adv}\|$ of the transport operator is used.

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\|L_{adv}\|} \ . \tag{12}$$

The optimality parameter α_{opt} is computed in the same way as in equation (9)

$$\alpha_{opt} := \coth(Pe) - \frac{1}{Pe},$$

but the element Peclet number now depends on the operator norms of the advection and diffusion differential operator

$$Pe := \frac{\| L_{adv} \| \cdot h_e}{\| L_{diff} \|}. \tag{13}$$

The choice of a suitable operator norm has a large influence on the quality of the solution. On the basis of the general definition [3] of the norm of continuity operators in (Euclidean) normed spaces we define the following operator norm. The differential operator has the form presented in (4)

$$L = \sum L_i = \sum A_i \frac{\partial}{\partial x_i}$$

then the operator norm is

$$\| L \| := \sqrt{\sum \| L_i \|^2} \tag{14}$$

where the norm of the operator component is calculated by

$$\| L_i \| := \left| \lambda_{max}(A_i) \right|$$

(15)

with $\lambda_{max}(A_i)$ the absolutely largest eigenvalue of the Matrix A_i.

This definition is consistent in all dimension, starting by the one dimensional scalar valued advective diffusive problem up to more dimensional and vector valued problems. This approach for the stabilization parameter leads to very good numerical results for a large number of simulation models. In chapter 3, two different models are presented and results are discussed.

3 Numerical examples

3.1 Macroscopic traffic modelling

Macroscopic traffic models are an essential tools to assess and control traffic flows on main highways. The traffic is described by the mean velocity V and the traffic density ρ. The used equations are very similar to the Navier-Stokes equations such as the macroscopic traffic model of Kerner and Konhäuser [2].

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -V \frac{\partial \rho}{\partial x} - \rho \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial t} &= -V \frac{\partial V}{\partial x} - \frac{c_0^2}{\rho} \frac{\partial \rho}{\partial x} + \frac{\mu}{\rho} \frac{\partial^2 V}{\partial x^2} + \frac{1}{\tau} \left(V_e - V \right) \end{aligned} \tag{16}$$

In contrast to fluid dynamics, the law of conservation of momentum is not valid in traffic dynamics. The term $(V_e - V)/\tau$ is called the adaptation term or relaxation term. It is assumed that the current velocity $V(x, t)$ is adapted to a prescribed equilibrium velocity V_e within a certain time τ. The equilibrium velocity V_e, which depends at least on the density, and the adaptation time are the most important parameters of all macroscopic traffic models. The model is characterized by a density dependent pressure parametrisation with constant velocity of propagation c_0^2 and due to an additional diffusion term, with constant viscosity μ which reflects some observations of viscous traffic flow in reality.

The above one-dimensional vector valued advection diffusion problem (16) can be expressed as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} - B\frac{\partial^2 U}{\partial x^2} - S = 0 \tag{17}$$

with

$$U = \begin{bmatrix} \rho \\ V \end{bmatrix} \quad , \quad A = \begin{bmatrix} V & \rho \\ \dfrac{c_0^2}{\rho} & V \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\mu}{\rho} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 \\ \dfrac{V_e - V}{\tau} \end{bmatrix} \quad .$$

Now we have the following operator-norms

$$\left\| A\frac{\partial}{\partial x} \right\| = |V| + |c_0| \quad \text{and} \quad \left\| B\frac{\partial^2}{\partial x^2} \right\| = \frac{\mu}{\rho}$$

as well as the element stabilization parameter

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{\Delta x}{|V| + |c_0|} \quad . \tag{18}$$

Here is $\alpha_{opt} := \coth(Pe) - \dfrac{1}{Pe}$, and

the element Peclet number can be determined by

$$Pe := \frac{\left(|V| + |c_0| \right) \cdot h}{\mu / \rho} \quad . \tag{19}$$

The complexity of the arising phenomena is demonstrated in an academic case example. A closed single lane is considered. It is assumed that the initial traffic state is in equilibrium state with a prescribed density ρ_e and a small local perturbation of the uniform density distribution. The properties of the jam formations are studied in detail depending on the density ρ_e.

Figure 1 shows some typical results of a numerical perturbation analysis for macroscopic traffic models. The equilibrium is stable for free traffic with $\rho_e<15$ *veh./km* and for dense traffic with $\rho_e>60$ *veh./km*. In the stable case the perturbations of the equilibrium state decrease with increasing time. The equilibrium is unstable for congested traffic. There are three different phenomena of congested traffic with respect to jam formations: simple jam, stop and go traffic and wide jam. The jams move backward with a velocity of about *-15±5 km/h*. The different phenomena of jam formation can also be observed in real traffic on highways.

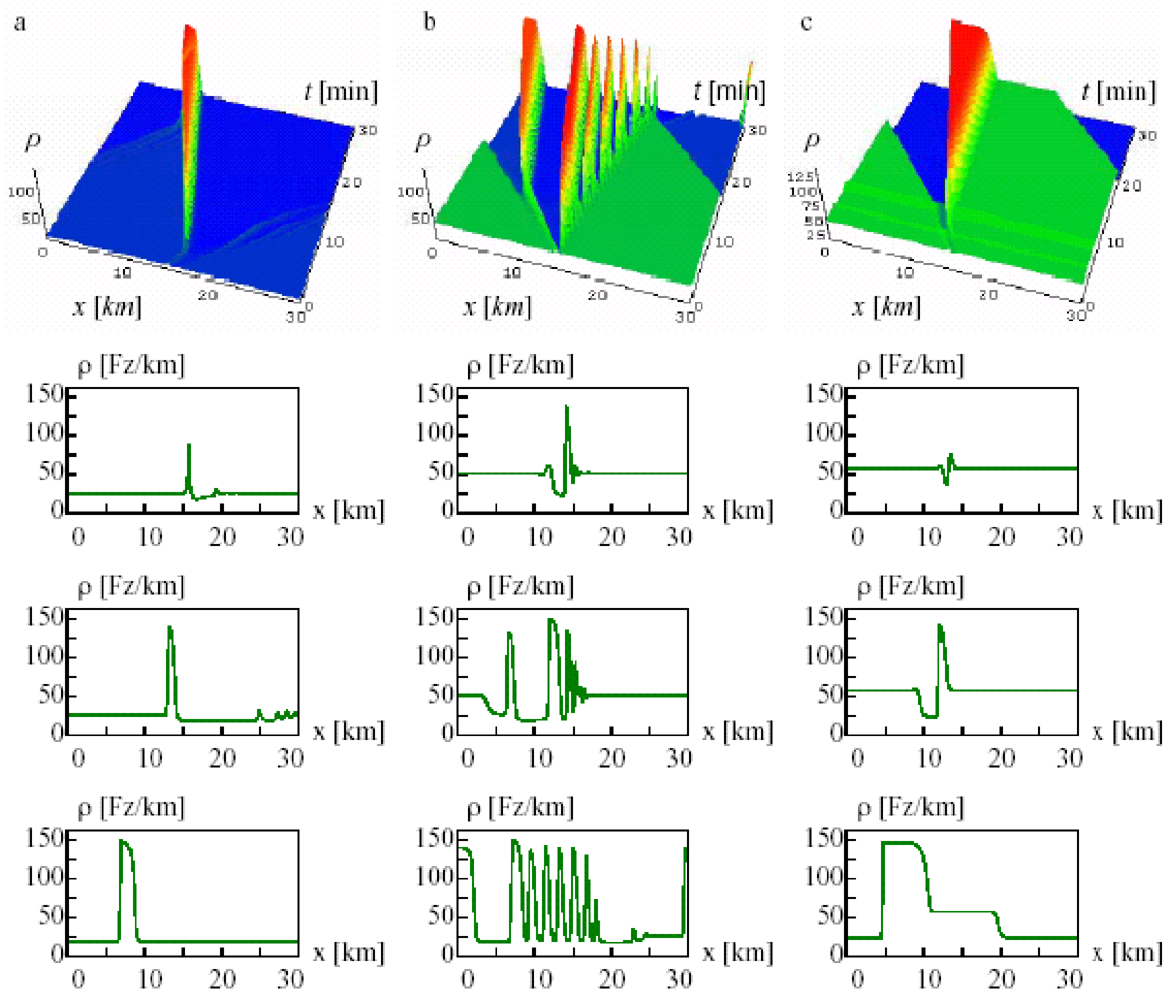

Figure 1: Density evolutions with ρ_e = 25 veh/km (a), ρ_e = 50 veh/km (b) and ρ_e = 56.25 veh/km (c) in space-time diagram and spatial density distributions at time t=3, 11, 30 min

3.2 Morphodynamic modeling in coastal engineering

The hydro- and morphodynamic processes in the nearshore area create highly complex phenomena. Suitable modeling tools are necessary for the assessment of natural developments in coastal areas as well as of impacts of human interventions in form of coastal protection buildings. The phenomena to be taken into account are wind waves, currents and sediment transport. These can be described by the following system of 9 time-depending partial differential equations [4].

$$
\begin{aligned}
\frac{\partial K_i}{\partial t} &= -\frac{\partial \sigma_a}{\partial x_i} + C_g \frac{K_j}{k}\left(\frac{\partial K_j}{\partial x_i} - \frac{\partial K_i}{\partial x_j}\right)\\
\frac{\partial \sigma}{\partial t} &= -\left(U_i + C_{g_i}\right)\frac{\partial \sigma}{\partial x_i} - k_{x_i}\frac{\partial u_i}{\partial t} + f \cdot \frac{\partial h}{\partial t}\\
\frac{\partial a}{\partial t} &= -\frac{1}{2a}\frac{\partial}{\partial x_i}\left(U_i + C_{E_i}\right) a^2 - \frac{S_{ij}}{\rho\, ga}\frac{\partial U_i}{\partial x_i} + \frac{U_i\left(T_i - T_i^B\right)}{\rho\, ga} + \frac{\epsilon_B}{\rho\, ga}\\
\\
\frac{\partial U_i}{\partial t} &= -U_j\frac{\partial U_i}{\partial x_j} - g\frac{\partial \overline{\eta}}{\partial x_i} - \frac{1}{\rho\, d}\frac{\partial S_{ij}}{\partial x_i} + \frac{1}{\rho\, d}\left(T_i - T_i^B\right)\\
\frac{\partial \overline{\eta}}{\partial t} &= -\frac{\partial U_j d}{\partial x_j}\\
\\
\frac{\partial C}{\partial t} &= -U_i\frac{\partial C}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\tau_i \frac{\partial C}{\partial x_i}\right) + S\\
q_i &= \int_{-h}^{\eta} U_i C\, dz + q_{b_i}\\
\frac{\partial h}{\partial t} &= -\frac{1}{1-n}\frac{\partial q_i}{\partial x_i}
\end{aligned}
\tag{20}
$$

Here the first equations describe the evolution of wind waves, with the wave number vector *K*, the angular frequency *σ* and the wave amplitude *a*. The second block are the shallow water equations with the current vector *U* and the mean water evaluation $\overline{\eta}$. The last equations describe the sediment transport and the changes of the sea bottom.

Each of these partial models for waves (W), current (U) and transport (S) can be transformed into the above simplified form by introduction of suitable differential operators.

For example, the second partial model, the two dimensional shallow water equation, can be expressed as

$$\frac{\partial U}{\partial t} + A_x\frac{\partial U}{\partial x} + A_y\frac{\partial U}{\partial y} - B_x\frac{\partial^2 U}{\partial x^2} - B_y\frac{\partial^2 U}{\partial y^2} = 0 \tag{21}$$

with

$$U = \begin{bmatrix} U_x \\ U_y \\ \overline{\eta} \end{bmatrix}, \quad A_x = \begin{bmatrix} U_x & 0 & g \\ 0 & U_x & 0 \\ d & 0 & U_x \end{bmatrix}, \quad A_y = \begin{bmatrix} U_y & 0 & 0 \\ 0 & U_y & g \\ 0 & d & U_y \end{bmatrix},$$

$$B_x = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the diffusion coeffision ϵ . Now we have the following operator-norms

$$\left\| L_{adv} \right\| = \sqrt{\left(|v_x| + \sqrt{g\cdot d}\right)^2 + \left(|v_y| + \sqrt{g\cdot d}\right)^2} \tag{22}$$

and

$$\left\| L_{diff} \right\| = 2\cdot\epsilon \tag{23}$$

as well as the element stabilization parameter

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\| L_{adv} \|} \quad \text{with} \quad \alpha_{opt} := \coth(Pe) - \frac{1}{Pe} \quad \text{and} \quad Pe := \frac{\| L_{adv} \| \cdot h_e}{\| L_{diff} \|} .$$

The presented model system has been applied to an investigation of morphodynamic processes in coastal areas. The south tip of the island Sylt, located in the German bay, is formed by wind waves and tidal currents. The Assessment of the bathymetric stability is a substantial pre-condition for coastal protection interferences by humans.
To specify the influence of wind waves on the dynamic stability of the system Sylt comparisons between morphodynamic simulation results with and without the influence of wind waves were made.
The model area covers the entire island of Sylt. The simulation period covers 10 mean tides, whereby only the last tidal period was evaluated in the analysis. For the wind waves a quasi-stable situation with waves from west, a single wave period of 5s and a wave height of 1m in deep water was used.

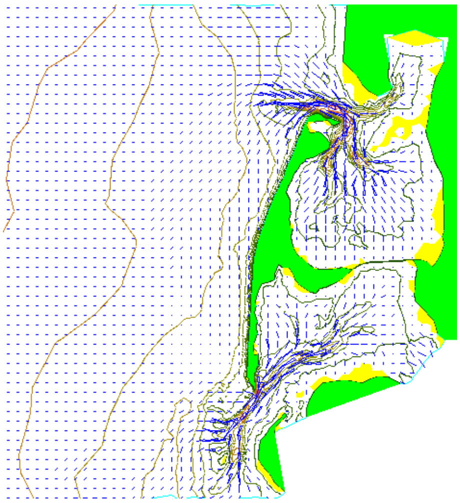
Figure 2: flow field

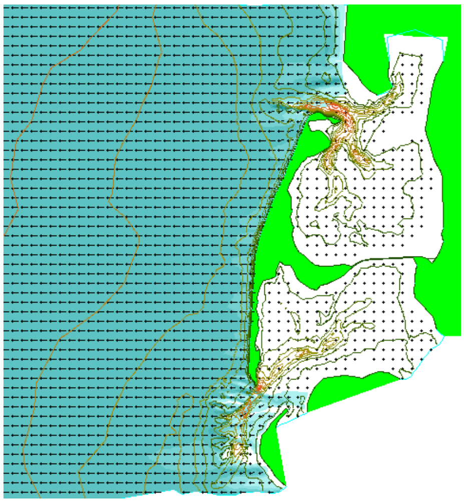
Figure 3: wave height and direction

The completely coupled model computation updates the depth in each calculation step. The presented depth changes are determined by taking the difference between the last two tidal periods. Red areas mark areas of erosion and blue those of sedimentation.

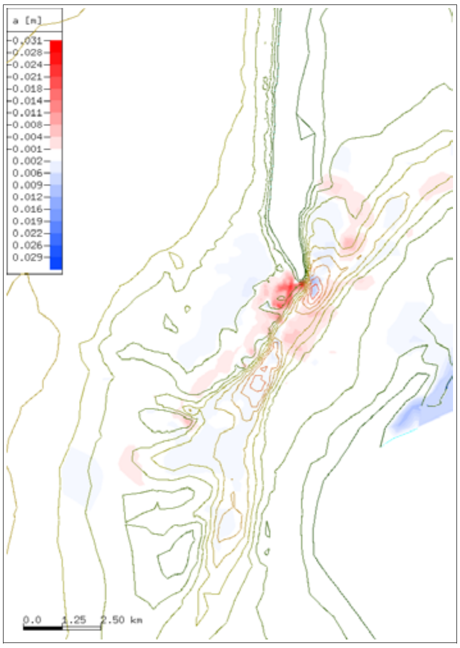

Figure 4 erosion and sedimentation only by tidal influence

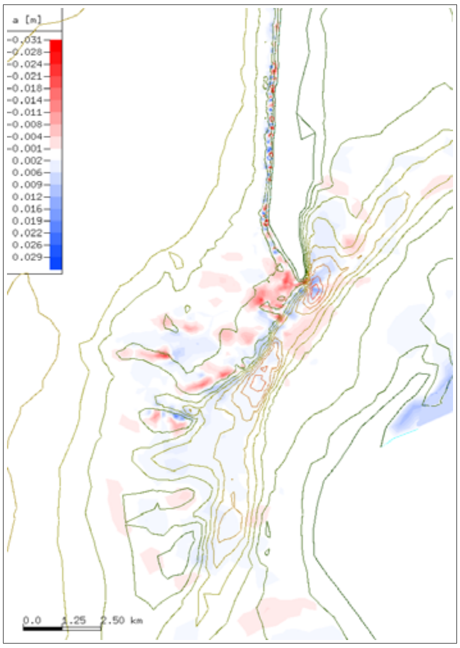

Figure 5 erosion and sedimentation by combined effects of waves and tide

The illustrations show clearly the significance of wind waves to the morphologic dynamic stability of the south tip of the Sylt island. If the waves are neglected and only tides are regarded as the dominant factor, wrong conclusions are likely to be drawn. A holistic view as well as a holistic modeling of all significant processes in the coastal area are necessary in order to capture the relevant processes.

4 Conclusions

A stabilized finite element procedure on the basis of a Galerkin/least-squares approximation for a wide range of applications was presented. A general rule is indicated for computing suitable element stabilization parameters using an operator norm of the differential operators and the element expansion. Numerical examples have shown that the new formulation successfully improves the stability.

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