Polyhedral Finite Element Approximation for Transport Problems

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Abstract

The solution of advection-dominated equations with the method of finite elements led to the development of stabilization techniques. In this paper we present that the generalized formulation of stabilized finite elements on the basis of a Galerkin/least-squares approximation [2], [4] can be transferred to finite element method based on arbitrary convex polyhedron [3]. A condition for this is the formulation in natural element coordinates. The choice of suitable stabilization parameters is often application-dependent and difficult. A general rule for computing suitable element stabilization parameters is outlined which uses the spectral radius of the differential operators and the element expansion.

Keywords: natural element coordinates, stabilized finite elements, convex cells

1. Introduction

Many phenomena in physics and technology relate to transport phenomena and the reaction of states and substances. Such physical and technical questions can often be represented with stationary or transient partial differential equations. The approximation of transport dominant problems with the method of finite differences or the method of finite elements frequently leads to instabilities of the approximated solution. To overcome most of the limitations in the Galerkin method by solving transport dominant problems, the stabilized finite element method based on a combination of the Galerkin and least-squares approach [2]. The choice of suitable stabilization parameters is difficult and often application-dependent. The generalized finite elements on the basis of convex polyhedrons make a higher flexibility possible with the generation of decompositions and simplify completely substantially adaptive refinements. If we would use these new elements to solve transport dominated problems a suitable stabilization is necessary.

2. Finite Elements on Convex Polyhedrons

A generalised finite element FE can be understood as a triple consisting of a geometrical basis GE, a set of degrees of freedom Θ and a set of interpolation functions Φ [1]:

$$FE := (GE, \Theta, \Phi)$$

The complete description of complex problems will be realised with a set of simple interpolation functions with unknown parameters for sub regions (finite elements) of an element decomposition. The solution of a differential equation can be approximated with the solution of corresponding algebraic system of equations. A degree of freedom is normally composed of a point (element of the geometrical basis), an interpolation function and a value. The corresponding interpolation function is described by natural element coordinates of the geometrical basis and the value of one is assigned to the corresponding degree of freedom.

2.1. Natural Element Coordinates

The formulation of a local coordinate system permits a uniform element formulation in the method of the finite elements. The description of the convex polyhedron Z by the Minkowsky product

$$Z := \{p : p = \lambda_1 e^1 + \lambda_2 e^2 + \ldots + \lambda_N e^N, \lambda_i \geqslant 0 \wedge \sum_i \lambda_i = 1\} \tag{1}$$

of its vertices E suggests to use the factors λ_i of the linear combination as element coordinates. If an m-dimensional convex polyhedron has $m+1$ linear independent vertices, the factors are unique and called barycentric coordinates. If a convex polyhedron consists of more than $m+1$ vertices, the factors are not unique. If the natural neighborhood coordinates introduced by Sibson [6] are restricted to the convex polyhedron, one receives unique natural element coordinates, which are related to the vertices of the convex polyhedron. The determination of the natural element coordinates of a point x concerning the convex polyhedron Z is based on the computation of the Voronoi diagram of second order concerning the vertices and the point x.

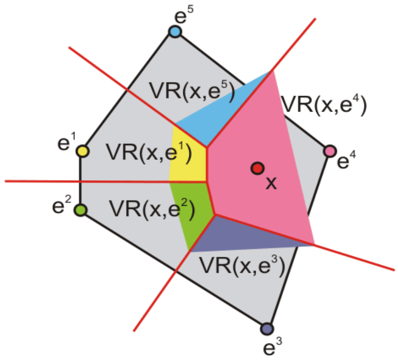

Figure 1: Voronoi decomposition of the convex polyhedron using sub-regions

Firstly, the Voronoi decomposition of first order of a convex polyhedron is determined by its vertices e^i. Each vertex of the convex polyhedron has its own Voronoi region. The Voronoi region of a vertex e^i is the set of all points p which has a smaller or equal distance to the vertex e^i as their distance to the remaining vertices e^j:

$$VR(e^i) := \{ p \in \mathbb{R}^n : d(p, e^i) \le d(p, e^j) \forall\, j \ne i \} . \quad (2)$$

The Voronoi region of second order of a convex polyhedron is determined concerning its vertices e^i and a point x of the convex polyhedron. A Voronoi region of second order is the set of points p, whose distance to the point x is smaller or equal their distance to a vertex e^i, if its distance to this vertex is smaller or equal their distance to the remaining vertices e^j:

$$VR(x, e^i) := \{ p \in \mathbb{R}^n : d(p, x) \le d(p, e^i) \le d(p, e^j) \forall\, j \ne i \} . \quad (3)$$

The natural element coordinates of the point x concerning the vertex e^i are determined over the Voronoi regions of second order (see Figure 1). Each Voronoi region of first or second order assigns itself a Lebesgue measure $\mu(VR(e^i))$ or $\mu(VR(x, e^i))$. This measure corresponds to the common surface area in the 2-dimensional Euclidean space. The ratio between the measure of the Voronoi region of second order of a vertex and the point x to measure of the Voronoi regions of first order of the point x concerning all vertices of the convex polyhedron is called the unique natural element coordinates

$$\lambda_i(x, e_i) := \frac{\mu(VR(x, e^i))}{\mu(VR(x))} . \quad (4)$$

3. Stabilized Finite Element Approximation

3.1. The Transport Problem

The following general transient problem shall be viewed. Let *Ω* represent the open bounded domain in $\mathbb{R}^n$ and *Γ* its boundary. Find a vector function $U : \Omega \to \mathbb{R}^m$ such that

$$\frac{\partial U}{\partial t} + L\,U + S = 0 \tag{5}$$

is valid, where *L* is a quasi-linear differential operator and *S* are source and sink terms. We assume that all necessary boundary and initial conditions which guarantee the existence of the solution are available.

The quasi-linear operator has the following form:

$$L \equiv A_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial}{\partial x_j}\right) \tag{6}$$

Here A_i is the i^{th} Euler Jacobian matrix and K_{ij} is the diffusivity matrix. Therefore, the operator *L* can be understood as sum of an advection operator L_{adv} and a diffusion operator L_{diff}:

$$L = L_{adv} + L_{diff} \tag{7}$$

each operator can be divided again into its local components with the following representation. As this can be shown for the transport operator:

$$L = \sum L_i = \sum A_i \frac{\partial}{\partial x_i} \tag{8}$$

In order to approximate the equation (5) with the finite element method the domain *Ω* is discretized into n_{el} finite elements Ω_e.

3.2. Stabilized Finite Element Approximation

The derivation of the stabilized finite element approximation is carried out via the combination of a standard Galerkin approximation and the least squares approximation. This can be described roughly, for the differential equation (5) as follows:

$$\int_{\Omega} \left(U_{,t} + L\,U + S\right) \cdot w \; d\Omega + \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e} \left(L \cdot w\right)\left(U_{,t} + L\,U + S\right) d\Omega^e = 0 \tag{9}$$

The first integral contains the Galerkin approximation and the second term contains the least-squares stabilization which is composed of the sum of integrals over the element interiors. This approximation is called semi-discrete GLS method. We use the following modified semi-discrete SUPG method, which is a predecessor to the GLS method.

$$\int_{\Omega} \left(U_{,t}+L\,U+S\right)\cdot w\; d\,\Omega+\sum_{e=1}^{n_{el}} \tau_e \int_{\Omega^e} \left(L_{adv}\cdot w\right)\left(U_{,t}^{G}+L\,U+S\right) d\,\Omega^e=0 \tag{10}$$

where $U_{,t}^{G}$ is determined by the standard Galerkin-method. The difference to the GLS is that rather than having L operating on the weighting space, only its advective part, L_{adv}, acts there.

The element stabilization parameter τ_e plays an important role for the stability and consistency of the approximation.

3.3. Stabilization Parameter

On the basis of the determination of an optimal stability parameter in the one-dimensional case the following formulation for multidimensional vector valued transport problem results

$$\tau_e:=\alpha_{opt}\frac{1}{2}\frac{h_e}{\rho(L_{adv})}\ . \tag{11}$$

with $\rho(L_{adv})$ is the spectral radius of the quasi-linear transport operator. The determination of the element expansion h_e is represented in Figure 2.

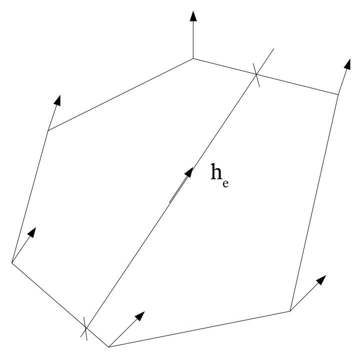

Figure 2: Computation of the element size

The optimality parameter α_{opt} is evaluated in the same way as in the one-dimensional case

$$\alpha_{opt}:=\coth(Pe)-\frac{1}{Pe}\ , \tag{12}$$

but the element Peclet number now depends on the spectral radii of the advection and diffusion differential operator

$$Pe:=\frac{\rho(L_{adv})\cdot h_e}{\rho(L_{diff})}\ . \tag{13}$$

The differential operator has the form presented in (6)

$$L = \sum L_i = \sum A_i \frac{\partial}{\partial x_i}$$

then the spectral radius of the operator is

$$\rho(L) := \sqrt{\sum \rho(L_i)^2} \tag{14}$$

where the spectral radius of the operator component is calculated by

$$\rho(L_i) := \left|\lambda_{max}(A_i)\right| \tag{15}$$

with $\lambda_{max}(A_i)$ the absolutely largest eigenvalue of the Matrix A_i.

This definition is consistent in all dimensions, starting by the one dimensional scalar valued advective diffusive problem up to more dimensional and vector valued problems.

4. Numerical Examples

The approximation of the instantaneous simplified shallow water equation in a rectangular basin is used as a test case.

$$\begin{aligned} \frac{\partial U_i}{\partial t} &= -U_j \frac{\partial U_i}{\partial x_j} - g \frac{\partial \bar{\eta}}{\partial x_i} + \frac{1}{\rho d}\left(T_i - T_i^B\right) \\ \frac{\partial \bar{\eta}}{\partial t} &= -\frac{\partial U_j d}{\partial x_j} \end{aligned} \tag{16}$$

where $\bar{\eta}$ is the mean water level U_i representing the depth integrated velocities in x- and y-direction, d is the mean water depth.

The rectangular basin have a steadily water inflow in the upper left corner and moves into the lower right corner. Obviously, this finite element decomposition does not consist of triangles or quadrangles, but it is a decomposition of any convex polygonal cells. The decomposition was generated by a Voronoi decomposition and consist of 2455 convex polygonal cells with different numbers of edges and 5067 degrees of freedom.

The Figure 3 shows the results of the stabilized finite element simulation after 8 and 150 minutes. The typical structure of a whirl appeared at the inflow side oft the basin and moved to the right side (150 minutes) to build a quasi stationary whirl. The perspective is from above.

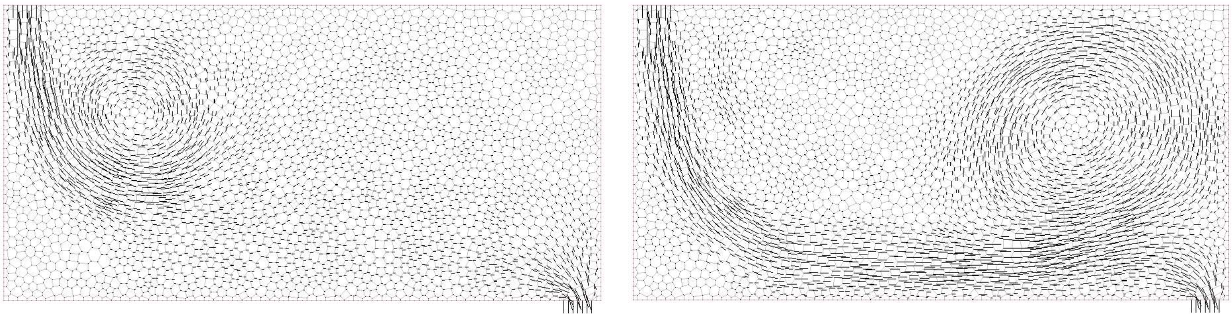

Figure 3: Finite element decomposition and current velocities field after 8 minutes and after 150 minutes

The developing flow has a smooth structure. With the presented implementation of a generalized finite element approximation comparison with different decompositions can be investigated.

5. Conclusion

A stabilized finite element procedure on the basis of a Galerkin / least-squares approximation for arbitrary convex polyhedrons was presented. A general rule is indicated for computing suitable element stabilization parameters using the spectral radius of the differential operators and the element expansion. A numerical example from fluid mechanics has shown that the new formulation successfully improves the stability.

References

[1] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, Classics in applied mathematics; 40, siam 2002, ISBN 0-89871-514-8

[2] T.J.R. Hughes, L.P. Franca, and G.M. Hulbert, A new finite element formulation for fluid dynamics: VIII. The Galerkin / least-squares method for advective-diffusive equations, Computer Methods in Applied Mechanics and Engineering, 73, (1989), 173-189.

[3] Milbradt, P.: Algorithmische Geometrie in der Bauinformatik. - Habilitationsschrift, Universität Hannover, 2001

[4] Milbradt, P.; A stabilized finite element method for systems of instationary advection-dominated equations in multi-dimensional domains; Fifth World Congress on Mechanics 2002 (Vienna)

[5] Schwöppe, A. and Milbradt, P.; A Class of Convex, Polygonal Bounded Finite Elements; Fifth World Congress on Mechanics 2002 (Vienna)

[6] Sibson, R.: A vector identity for the Dirichlet tessellation. In: Mathematical Proceedings of the

Cambridge Philosophical Society Bd. 87, 1980, S. 151-155