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# UNDERSTANDING THE ASPECT OF FUZZINESS IN INTERPOLATION METHODS

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**Abstract.** *Fuzzy functions are suitable to deal with uncertainties and fuzziness in a closed form maintaining the informational content. This paper tries to understand, elaborate, and explain the problem of interpolating crisp and fuzzy data using continuous fuzzy valued functions. Two main issues are addressed here. The first covers how the fuzziness, induced by the reduction and deficit of information i.e. the discontinuity of the interpolated points, can be evaluated considering the used interpolation method and the density of the data. The second issue deals with the need to differentiate between impreciseness and hence fuzziness only in the interpolated quantity, impreciseness only in the location of the interpolated points and impreciseness in both the quantity and the location.*

*In this paper, a brief background of the concept of fuzzy numbers and of fuzzy functions is presented. The numerical side of computing with fuzzy numbers is concisely demonstrated. The problem of fuzzy polynomial interpolation, the interpolation on meshes and mesh free fuzzy interpolation is investigated. The integration of the previously noted uncertainty into a coherent fuzzy valued function is discussed. Several sets of artificial and original measured data are used to examine the mentioned fuzzy interpolations.*

## 1 INTRODUCTION

Interpolation methods are used to construct idealized real valued continuous functions from distributed data. Due to the in reality inevitably point wise discrete and imprecise observations of the quantity in interest, constructing real valued continuous functions is not adequate to represent the complete information on the observed data. The fact, that the interpolated data points are scattered and discrete, causes a reduction of the actual continuous function. This will necessarily introduce some kind of uncertainty and fuzziness caused by the information deficit. In the case of inaccurate measurements this fuzziness is genuine. It is thus necessary to consider this uncertainty and fuzziness in order to get an interpolating function, that incorporates all the available information.

Fuzzy functions are suitable to deal with the mentioned uncertainties and fuzziness in a closed form maintaining the informational content. Two main issues must be addressed. The first covers how the fuzziness, induced by the reduction and deficiency of information i.e. the discontinuity of the interpolated points, can be evaluated considering the used interpolation method and the density of the data. The second issue deals with the need to differentiate between impreciseness and hence fuzziness only in the interpolated quantity, impreciseness only in the location of the interpolated points and impreciseness in both the quantity and the location.

In the next section a brief background of the concept of fuzzy numbers and of fuzzy functions is presented. After that in section 3 the numerical side of computing with fuzzy numbers is concisely demonstrated. The integration of the previously noted uncertainty into a coherent fuzzy valued function is discussed in section 4. In section 5 sets of artificial and original measured data are used to examine the fuzzy Lagrange polynomial interpolation, the mesh free fuzzy Shepard interpolation, the fuzzy piecewise linear interpolation and the fuzzy piecewise bilinear interpolation on a grid.

## 2 FUZZY NUMBERS AND FUZZY FUNCTIONS

A fuzzy number $\widetilde{x}$ is a special case of a fuzzy set with specific requirements on the membership function $\mu(x)$. A definition of the fuzzy number is given as follows.

**Definition 2.1 (Fuzzy Number)** *A fuzzy number $\widetilde{x}$ is a fuzzy set with the membership function $\mu(x) : \mathbb{R} \to [0,1]$ that is piecewise continuous such that:*

1. *For every pair $\alpha_1$ and $\alpha_2$ in $[0,1]$, whenever $\alpha_1 < \alpha_2$ then $\mathbb{R} \supseteq A_{\alpha_1} \supseteq A_{\alpha_2}$, with $A_{\alpha_j} = \{x \in \mathbb{R} : \mu(x) > \alpha_j\}$ the $\alpha$-Cut set. This means, the fuzzy set is convex.*
2. $\sup\limits_{x\in\mathbb{R}} \mu(x) = 1$. *This means, the fuzzy set is normal.*
3. *There is exactly one $x \in \mathbb{R}$ with $\mu(x) = 1$ .*

The set of all fuzzy numbers of $\mathbb{R}$ is denoted by $\widetilde{\mathbb{R}}$ and called the fuzzy space. A crisp real number $x \in \mathbb{R}$ is regarded as a single tone set

$$\{(\xi, \mu(\xi))|\xi \in \mathbf{R}, \mu(\xi) \in \mathbf{R} \to [0,1]\}, \tag{1}$$

where

$$\mu(\xi) = \begin{cases} 1 & \text{for } \xi = x \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

which is a fuzzy number. Therefore, the set of crisp real numbers $\mathbb{R}$ can be seen as a subset of $\widetilde{\mathbb{R}}$ ($\mathbb{R} \subset \widetilde{\mathbb{R}}$).

Now the definition of a fuzzy function $\widetilde{f}$ can be given as follows.

**Definition 2.2 (Fuzzy Function)** *A fuzzy function is a relationship between two subsets $\widetilde{X}$ and $\widetilde{Y}$ of the fuzzy space $\widetilde{\mathbb{R}}$ that associats each element $\widetilde{x}$ of the first subset $\widetilde{X}$ with only one element of the second set $\widetilde{Y}$. $\widetilde{f}(\widetilde{x})$ is used to refer to the associated element $\widetilde{y} = \widetilde{f}(\widetilde{x})$ of the second set $\widetilde{Y}$ and is called the "fuzzy image of $\widetilde{x}$". This relationship is written as*

$$\widetilde{f}(\widetilde{x}) : \widetilde{X} \rightarrow \widetilde{Y} : \widetilde{y} = \widetilde{f}(\widetilde{x}) \quad . \tag{3}$$

A classical real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a special case of the fuzzy one, in which the domain and the range of the function are subsets of the fuzzy space $\widetilde{\mathbb{R}}$ with only single tone fuzzy numbers. Defining the fuzzy function in an arithmetic sense is accomplished by the extension principle of Zadeh, which is the subject of the next section.

## 3 COMPUTING WITH FUZZY NUMBERS

The extension principle, as proposed by Zadeh, is the basic concept providing a method to extend the arithmetic notion of real functions to fuzzy functions. This allows the definition of mathematical operations on fuzzy numbers in a similar way as in the real numbers space. The extension principle is stated as in the following.

**Proposition 3.1 (The Extension Principle of ZADEH)** *Let $X_1 \times X_2 \times ... \times X_n$, where $X_i \subseteq \mathbb{R}$ with $i = 1, 2, ..., n$, be a product set and $f$ a functional mapping of the form*

$$f : X_1 \times X_2 \times ... \times X_n \rightarrow Y \quad , \tag{4}$$

*which associates each element $(x_1, x_2, ..., x_n)$ of the product set to one element $y = f(x_1, x_2, ..., x_n)$ of the set $Y \subseteq \mathbb{R}$. Now, let $\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n$ be $n$ fuzzy numbers, defined as*

$$\widetilde{x}_i = \{(x_i, \mu_{\widetilde{x}_i}(x_i)) | x_i \in X_i \subseteq \mathbb{R}, \mu_{\widetilde{x}_i}(x_i) : X_i \rightarrow [0, 1]\} \tag{5}$$

*with $i = 1, 2, ..., n$ . Then the fuzzy number*

$$\widetilde{y} = \widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n) = \{(y, \mu_{\widetilde{y}}(y)) | y \in Y \subseteq \mathbb{R}, \mu_{\widetilde{y}}(y) : Y \rightarrow [0, 1]\} \tag{6}$$

*is the fuzzy image of $(\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n)$ resulting from extending the real functional mapping $f$ to a fuzzy functional mapping $\widetilde{f}$ and has the membership function $\mu_{\widetilde{y}}(y)$ defined by*

$$\mu_{\widetilde{y}}(y) = \begin{cases} \sup\limits_{y = f(x_1, x_2, ..., x_n)} \min\{\mu_{\widetilde{x}_1}(x_1), \mu_{\widetilde{x}_2}(x_2), ..., \mu_{\widetilde{x}_n}(x_n)\} & \textit{if} \quad \exists y = f(x_1, x_2, ..., x_n) \\ 0 & \textit{otherwise.} \end{cases} \tag{7}$$

Evaluating the extension principle is a subject of a long and diversifying discussion, which will be omitted here in favour of being concise. For more information about the basics of fuzzy set theory and the arithmetic of fuzzy numbers the references [1] and [2] are recommended.

## 4 FUZZY INTERPOLATION

Fuzzy interpolation methods can be understood as procedures for interpolating data with genuine fuzziness as well as procedures for interpolating crisp and fuzzy data quantifying the fuzziness induced by information deficiency. In other words, it is distinguished between two situations. The first situation is the interpolation of fuzzy data, in which the fuzziness is a non separable part of the sampling points. The second situation is the fuzzy interpolation of either crisp or fuzzy data quantifying the fuzziness induced by the reduction of the actual state of the interpolated analytical or empirical function.

Considering a set of crisp data, such that at various points $x_i$ there is a crisp information $y_i$ with $i = 1, \cdots, n$, the interpolation of such discrete crisp data in terms of relatively simple functions is well-grounded. The interpolation methods used are generally based on the simple form of an interpolating function

$$\overline{f} : \mathbb{R} \to \mathbb{R} : \overline{f}(x) = \sum_{i=1}^{n} y_i \cdot \phi_i(x) \tag{8}$$

where the basis function $\phi_i(x) : \mathbb{R} \to \mathbb{R}$ satisfies the interpolation condition:

$$\phi_i(x_k) = \begin{cases} 1 & \text{for } k = i \\ 0 & \text{for } k \neq i \end{cases} \tag{9}$$

A very simple example can be given in fitting a linear function to two data points $(x_1, y_1)$ and $(x_2, y_2)$, see Figure 3, which are sample values of some function, wether evaluated analytically or measured empirically.

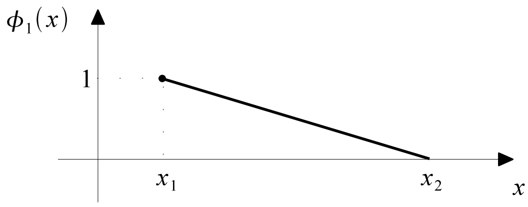


Figure 1: Basis function $\phi_1$ .

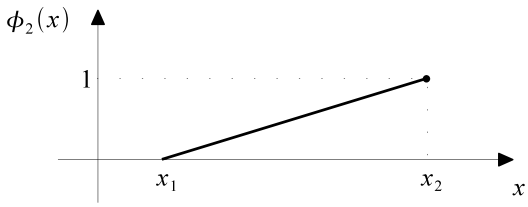


Figure 2: Basis function $\phi_2$ .

The basis functions that build the desired linear function are

$$\phi_1(x) \ := \ \frac{x_2 - x}{x_2 - x_1} \tag{10}$$

and

$$\phi_2(x) \ := \ \frac{x - x_1}{x_2 - x_1} \ , \tag{11}$$

see Figure 1 and Figure 2, and the graph of the linear interpolating function is shown in Figure 3 .

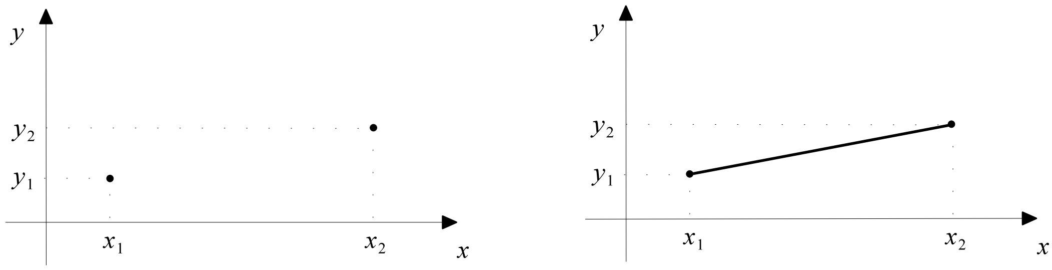


Figure 3: Linear interpolating function.

A general and systematic view of the interpolation methods that quantify the fuzziness induced by interpolating discrete and distributed data and cover interpolating data with genuine fuzziness is required.

### 4.1 Genuine fuzziness in the data

Here, the interpolation problem of fuzzy data in the general case is stated first. Then three essential special cases are discussed and demonstrated. The first case is interpolating data with fuzzy quantities at a crisp locations. The second case is interpolating data with crisp quantities at fuzzy locations. The third case is interpolating data with fuzzy quantities and fuzzy locations. At last the general case of interpolating fuzzy data is discussed.

**Problem 4.1 (The problem of interpolating data with genuine fuzziness)** *Let $\widetilde{x}_1, \cdots, \widetilde{x}_n$ be $n$ fuzzy points in $\widetilde{\mathbb{R}}^m$. A fuzzy number $\widetilde{y}_i$ in $\widetilde{\mathbb{R}}$ is associated to each $\widetilde{x}_i$ with $i = 1, \cdots, n$. The pairs $(\tilde{x}_i, \tilde{y}_i) \in \widetilde{\mathbb{R}}^m \times \widetilde{\mathbb{R}}$ are pairwise unique. Constructing an interpolating function $\widetilde{\widetilde{f}} : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ mapping $\widetilde{\mathbb{R}}^m$ to $\widetilde{\mathbb{R}}$ such that:*

1. *and $\widetilde{\widetilde{f}}(\tilde{x}_i) = \tilde{y}_i$ for all $i = 1, \cdots, n$,*
2. *$\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}}^m \to \widetilde{\mathbb{R}}$ is a continuous function.*

*is the problem to be sovled.*

In a similar way as in interpolating crisp data the sought function can then take the form

$$\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}}^m \to \widetilde{\mathbb{R}} : \widetilde{\widetilde{f}}(\widetilde{x}) = \sum_{i=1}^{n} \widetilde{y}_i \cdot \widetilde{\phi}_i(\widetilde{x}) \quad . \tag{12}$$

where the $\widetilde{\phi}_i : \widetilde{\mathbb{R}}^m \to \widetilde{\mathbb{R}}$ with $i = 1, \cdots, n$ are fuzzy functions that satisfy the interpolation condition

$$\widetilde{\phi}_i(\widetilde{x}_k) = \begin{cases} 1 & \text{for} \quad k = i \\ 0 & \text{for} \quad k \neq i \end{cases} \tag{13}$$

Different ways of defining $\widetilde{\phi}_i(\widetilde{x})$ give different interpolation methods, which will be examined later on. To simplify matters the one dimensional linear interpolation is considered as an introductory example in the next four cases.

The generalization of problem 4.1 to cover interpolating functions of the form $\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}}^m \to \widetilde{\mathbb{R}}^l$ mapping $\widetilde{\mathbb{R}}^m$ to $\widetilde{\mathbb{R}}^l$ is straight forward. The fuzzy number $\widetilde{y}_i$ is replaced with a fuzzy point in $\widetilde{\mathbb{R}}^l$.

### 4.1.1 Case 1: Interpolating data with fuzzy quantities at crisp locations.

In the case of interpolating data with fuzzy quantities $\widetilde{y}_i$ at crisp locations $x_i$ with $i = 1, ..., n$, which are sample values of some analytical fuzzy function or of some fuzzy empirically measured data, the basis functions $\phi_i : \mathbb{R} \to \mathbb{R}$ with $i = 0, ..., n$ can take the form of real functions as special cases of the fuzzy functions.

The interpolating function $\widetilde{\widetilde{f}} : \mathbb{R} \to \widetilde{\mathbb{R}}$ is given in a general form as

$$\widetilde{\widetilde{f}}(x) = \sum_{i=1}^{n} \widetilde{y}_i \cdot \phi_i(x) \quad . \tag{14}$$

For the fuzzy linear interpolating function of the data points $(x_1, \widetilde{y}_1)$ and $(x_2, \widetilde{y}_2)$, see Figure 4 . The basis functions $\phi_1(x)$ and $\phi_2(x)$ are given by

$$\phi_1(x) := \frac{x_2 - x}{x_2 - x_1} \tag{15}$$

and

$$\phi_2(x) := \frac{x - x_1}{x_2 - x_1} , \tag{16}$$

and depend only on the crisp location of the sampling points. For the graphs of the functions $\phi_1(x)$ and $\phi_2(x)$ see Figure 1 and Figure 2, respectively.

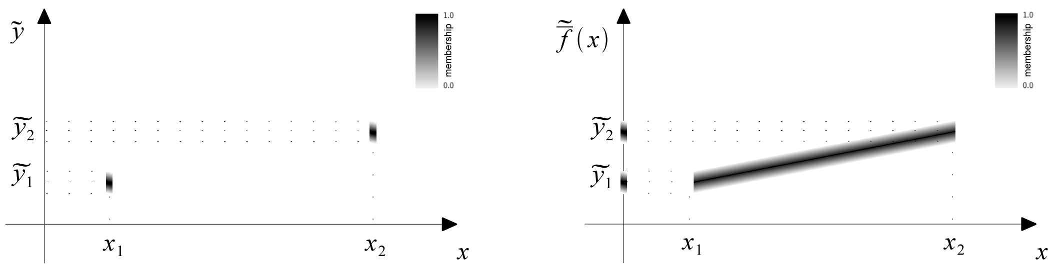


Figure 4: Fuzzy linear interpolating function.

The graph of the resulted fuzzy interpolating function is to be seen in Figure 4 .

### 4.1.2 Case 2: Interpolating data with crisp quantities at fuzzy locations.

Interpolating sample values of some analytical fuzzy function or of some empirically measured fuzzy data with crisp quantities $y_i$ at fuzzy locations $\widetilde{x}_i$ with $i = 1, ..., n$ can be realized by the general interpolating function $\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}$ and is given as

$$\widetilde{\widetilde{f}}(\widetilde{x}) \ = \sum_{i=1}^{n} y_i \cdot \widetilde{\phi}_i(\widetilde{x}) \quad . \tag{17}$$

The basis functions $\widetilde{\phi}_i : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}$ with $i = 0, ..., n$ are fuzzy functions.

The fuzzy interpolating function in Eq. 17 fulfills the requirements on the sought fuzzy interpolating function stated in problem 4.1. But the subject of interest of interpolation methods is normally interpolating unknown quantities at an exactly known positions. Therefore, for the practical application of a fuzzy interpolation method the fuzzy function in Eq. 17 can take the form $\widetilde{\widetilde{f}} : \mathbb{R} \to \widetilde{\mathbb{R}}$ and is given as

$$\widetilde{\widetilde{f}}(x) \ = \sum_{i=1}^{n} y_i \cdot \widetilde{\phi}_i(x) \quad , \tag{18}$$

and the associated basis functions are hence fuzzy functions of the form $\widetilde{\phi}_i : \mathbb{R} \to \widetilde{\mathbb{R}}$ still depending on the fuzzy locations $\widetilde{x}_i$ of the sampling points.

It is important to mention that this interpolating function does not ensure the reproduction of the sampling quantities exactly at the sampling position. However, it is a special case of the Eq. 17 and therefore it still fulfills the first requirement of the problem 4.1. The general form of Eq. 17 will be revisited in the next two cases, especially in the case of morphing fuzzy numbers.

For the fuzzy linear interpolating function $\widetilde{\widetilde{f}}$ of $(\widetilde{x}_1, y_1)$ and $(\widetilde{x}_2, y_2)$, see Figure 7, the fuzzy basis functions $\widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(x)$ are given by Eq. 19 and Eq 20, respectively.

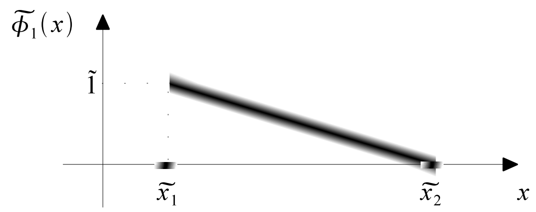


Figure 5: Basis function $\widetilde{\phi}_1$ .

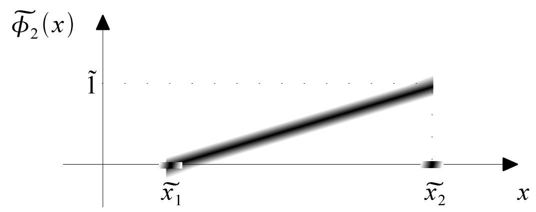


Figure 6: Basis function $\widetilde{\phi}_2$ .

$$\widetilde{\phi}_1(x) \ := \ \frac{\widetilde{x}_2 - x}{\widetilde{x}_2 - \widetilde{x}_1} \tag{19}$$

$$\widetilde{\phi}_2(x) \ := \ \frac{x - \widetilde{x}_1}{\widetilde{x}_2 - \widetilde{x}_1} \tag{20}$$

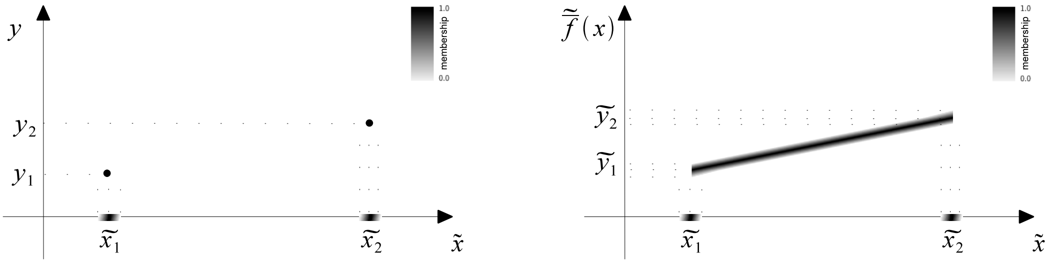


Figure 7: Fuzzy linear interpolating function.

Figure 5 and Figure 6 show the graph of the fuzzy basis functions $\widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(x)$, respectively. Figure 7 shows the graph of the resulted linear fuzzy interpolating function.

#### 4.1.3 Case 3: Interpolating data with fuzzy quantities at fuzzy locations.

Interpolating data with fuzzy quantities $\widetilde{y}_i$ at fuzzy locations $\widetilde{x}_i$ with $i = 1, ..., n$ can be done, as in the previous case, by the general interpolating function $\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}} \rightarrow \widetilde{\mathbb{R}}$ given as

$$\widetilde{\widetilde{f}}(\widetilde{x}) \;= \sum_{i=1}^{n} \widetilde{y}_i \cdot \widetilde{\phi}_i(\widetilde{x}) \quad . \tag{21}$$

with the basis functions of the form $\widetilde{\phi}_i : \widetilde{\mathbb{R}} \rightarrow \widetilde{\mathbb{R}}$ with $i = 0, ..., n$.

As in the previous case this interpolating function fullfils the requirements stated in problem 4.1. However, a special case of the fuzzy function in Eq. 21 will be used. This special fucntion takes the form $\widetilde{\widetilde{f}} : \mathbb{R} \rightarrow \widetilde{\mathbb{R}}$ and is given as

$$\widetilde{\widetilde{f}}(x) \;= \sum_{i=1}^{n} \widetilde{y}_i \cdot \widetilde{\phi}_i(x) \tag{22}$$

The basis functions $\widetilde{\phi}_i(x)$ with $i = 1, ..., n$, still depend on the fuzzy locations of the sampling points, are fuzzy functions of the form of $\widetilde{\phi}_i : \mathbb{R} \rightarrow \widetilde{\mathbb{R}}$. Again, it is important to note that this interpolating function in the special case does not ensure the reproduction of the sampling quantities exactly at the sampling position. Reproducing the sampling quantities is a subject of the next case.

The linear interpolating function $\widetilde{\widetilde{f}}$ of $(\widetilde{x}_1, \widetilde{y}_1)$ and $(\widetilde{x}_2, \widetilde{y}_2)$, see Figure 8, is constructed using the fuzzy basis functions $\widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(x)$ given in Eq. 23 and Eq. 24, respectively. For the graphs of the basis functions see Figure 5 and Figure 6 .

$$\widetilde{\phi}_1(x) \;:=\; \frac{\widetilde{x}_2 - x}{\widetilde{x}_2 - \widetilde{x}_1} \tag{23}$$

$$\widetilde{\phi}_2(x) \;:=\; \frac{x - \widetilde{x}_1}{\widetilde{x}_2 - \widetilde{x}_1} \tag{24}$$

See Figure 8 for the graph of this function.

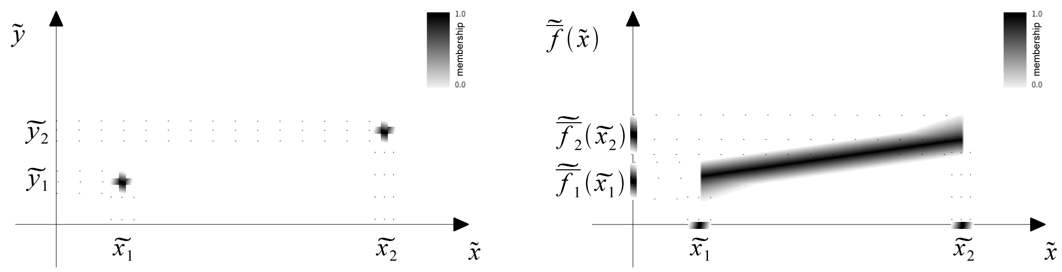


Figure 8: Fuzzy linear interpolating function.

#### 4.1.4 Case 4: The general case and morphing fuzzy numbers

In case 2 and case 3 the general fuzzy interpolating functions in Eq. 17 and Eq. 21 were avoided for practical reasons and substituted by the special fuzzy functions in Eq. 18 and Eq. 22 , respectively.

Generally speaking the fuzzy variable $\widetilde{x}$ in Eq. 17 and Eq. 21 can take values with unrestricted fuzziness in $\widetilde{\mathbb{R}}^m$. Despite the fact that this is mathematically justified, it does not prove any practical use. A fuzzy morphing space offers a reasonable restriction of the domain of the variable $\widetilde{x}$ . One can use a fuzzy morphing function

$$\widetilde{x} : \widetilde{\mathbb{R}}^m \times \widetilde{\mathbb{R}}^m \times \mathbb{R} \to \widetilde{\mathbb{R}}^m \tag{25}$$

which restrict the domain of $\widetilde{x}$ to a subset of $\widetilde{\mathbb{R}}^m$ . The interpolating fuzzy function then takes the form of Eq. 17 or Eq. 21 and fulfills the requirements of problem 4.1 in reproducing the sampling quantities exactly at the sampling points.

### 4.2 Fuzziness induced by deficiency of information

Interpolation methods are used to construct idealized real valued continuous functions from distributed data. Due to the in reality inevitably point wise discrete and imprecise observations of the quantity in interest, constructing real valued continuous functions is not adequate to represent the complete information on the observed data. Moreover, the fact that the sampling points are scattered and discrete, causes a reduction of the actual continuous function. This will necessarily introduce some kind of uncertainty caused by the deficiency of information.

The set of data that makes a specific interpolating function is regarded as an incomplete information item. Interpolation methods are used to get a totally unknown piece of information from this incompletely known information item. Thus, the values of the interpolating function are uncertain. In order to reduce this uncertainty the data set can be interpolated using a fuzzy interpolating function.

In this section the problem 4.1 is at first extended by an extra requirement to give the problem of interpolating data under information deficiency. Next an interpolating function for quantifying the fuzziness induced by the deficiency of information is suggested. Then two cases are discussed. The first case is interpolating data with crisp quantities at crisp locations. The second case is interpolating data with fuzzy quantities at crisp locations.

**Problem 4.2 (Interpolating data under information deficiency)** *Let $\widetilde{x}_1, \cdots, \widetilde{x}_n$ be $n$ fuzzy points in $\widetilde{\mathbb{R}}^m$. A fuzzy number $\widetilde{y}_i$ in $\widetilde{\mathbb{R}}$ is associated to each $\widetilde{x}_i$ with $i = 1, \cdots, n$. The pairs $(\tilde{x}_i, \tilde{y}_i) \in \widetilde{\mathbb{R}}^m \times \widetilde{\mathbb{R}}$ are pairwise unique. Constructing an interpolating function $\widetilde{\widetilde{f}} : \widetilde{\mathbb{R}}^m \to \widetilde{\mathbb{R}}$ mapping $\widetilde{\mathbb{R}}^m$ to $\widetilde{\mathbb{R}}$ such that:*

1. $\widetilde{\overline{f}}(\tilde{x}_i) = \tilde{y}_i$ *for all* $i = 1, \cdots, n$ *,*

2. $\widetilde{\overline{f}} : \widetilde{\mathbb{R}}^m \rightarrow \widetilde{\mathbb{R}}$ *is a continuous function,*

3. *and* $\widetilde{\overline{f}}(\widetilde{x})$ *quantifies the fuzziness induced by the distributed and discrete state of the data set*

*is the problem to be sovled.*

#### 4.2.1 Quantification of fuzziness

Quantifying the fuzziness induced by the discrete and distributed state of the sampling points will be demonstrated first. Let $x_1$ and $x_2$ be two pieces of information, each representing a specific location on the $x$ axis, see Figure 9. Considering the region between $x_1$ and $x_2$ as unknown, we could linearly interpolate the unknown location $x$ ($x_1 \leq x \leq x_2$) in a trivial way.

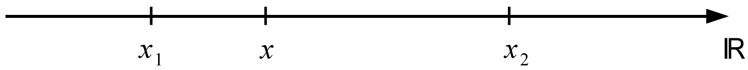


Figure 9: Coordinate interpolation.

The interpolating function is given as

$$\overline{x} = \sum_{i=1}^{2} x_i \cdot \phi_i(x) \quad , \tag{26}$$

where $\phi_i(x)$ are linear basis functions similar to those given in the last section in Eq. 10 and Eq. 11 for $\phi_1(x)$ and $\phi_2(x)$, respectively.

Nevertheless, having only $x_1$, $x_2$ and the interpolation method as an information item to describe the whole interval $[x_1, x_2]$ makes this information incomplete and induces an amount of uncertainty.

Quantifying the fuzziness here can be done by describing the interpolation position using a fuzzy function of the form

$$\widetilde{x}(x) : \mathbb{R} \rightarrow \widetilde{\mathbb{R}} : \widetilde{x} = \{(\xi, \mu(\xi)) | \xi \in \mathbf{R}, \mu(\xi) \in [x_1, x_2] \rightarrow [0, 1]\} \tag{27}$$

This function maps the coordinate $x$ of the interpolation position to a fuzzy number $\widetilde{x}$ that quantifies the fuzziness. The membership function $\mu(\xi)$ is given in a quasi-LR-representation, see Figure 10. That means the function is divided into a left function and a right function and given as

$$\mu(\xi) = \begin{cases} L(\frac{x-\xi}{x-x_1}) & \text{for } \; x_1 \leq \xi < x \\ 1 & \text{for } \quad \xi = x \\ R(\frac{\xi-x}{x_2-x}) & \text{for } \; x < \xi \leq x_2 \end{cases} \tag{28}$$

The left function $L$ is given as

$$L(\bar{\xi}) := (1 - \bar{\xi})^{\alpha_L}(1 - \phi_L) + \phi_L \;\; ; \;\; \alpha_L := \frac{1 - \phi_L}{\phi_L} \;\; ; \;\; \bar{\xi} = \frac{x - \xi}{x - x_1} \tag{29}$$

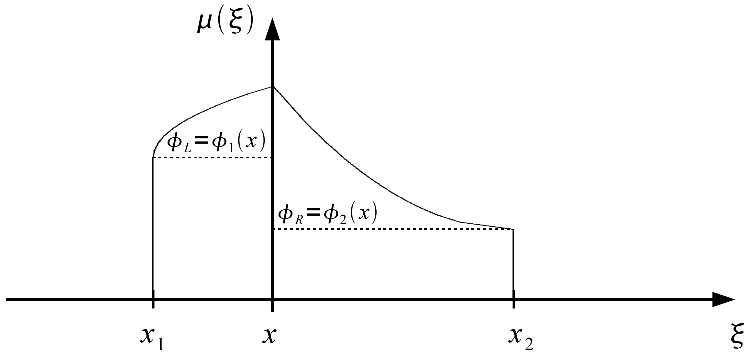


Figure 10: Quasi-LR-representation of a fuzzy number.

and the right function $R$ is given as

$$R(\bar{\xi}) := (1-\bar{\xi})^{\alpha_R}(1-\phi_R) + \phi_R \ ; \ \ \alpha_R := \frac{1-\phi_R}{\phi_R} \ \ ; \ \ \bar{\xi} = \frac{\xi - x}{x_2 - x} \tag{30}$$

The values of $\phi_L$ and $\phi_R$ are given by the basis functions $\phi_1$ and $\phi_2$, respectively.

Using the suggested quantification of the fuzziness in the coordinate $x$ between $x_1 = 0$ and $x_2 = 1$ gives the fuzzy numbers shown in Figure 11 at $x = 0.0$ ,$x = 0.25$ ,$x = 0.5$ ,$x = 0.75$ and $x = 1.0$.

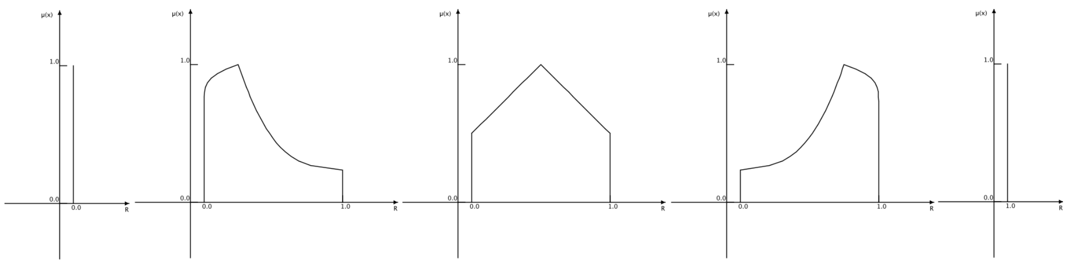

Figure 11: The interpolated fuzzy numbers at $x = 0.0$ ,$x = 0.25$ ,$x = 0.5$ ,$x = 0.75$ and $x = 1.0$ from left to right respectively.

The choice of the method to interpolate specific data is a subjective decision and depends on the expert opinion on which interpolating function best suits the interpolated data set. Additional information might be prescribed, such as the slope at given points, the data density and other characteristics of the data set.

The membership function of the constructed fuzzy number can be modified by a Characteristic-Density Factor. This factor expresses the information content of the association between the characteristics of the data set under study and the density of the data. This association is essential, because having only the density of the data does not justify the attempt to modify the membership function. Only in connection with the characteristic of the data set is this attempt plausible.

The Characteristic-Density Factor influences the quantification of the fuzziness. This influence can be demonstrated in decreasing or increasing the precision without affecting the uncertainty about the modified information value. This effect can be done by using a fuzzy modification operator on the fuzzy number representing the fuzzy value. In Eq. 31 the fuzzy modification operator $mod$ is applied on the membership function $\mu(\xi)$ of a fuzzy number

$$mod[\mu(\xi)] = \mu^c(\xi) \ , \tag{31}$$

where $c$ is the Characteristic-Density factor.

This Characteristic-Density factor $c$ could be given as

$$c = density^{|grad\, z(x,y)|} \ , \tag{32}$$

in case the gradient of the interpolated function is a-priori known or can be obtained. A suggested density function is given as

$$density = \sum_{i=1}^{n} D_i(p) \ , \tag{33}$$

where

$$D_i(p) = e^{-distance(p,p_i)^2} \tag{34}$$

and $p_i$ are the sampling points of the data set.

Describing the interpolation position x as a fuzzy number $\widetilde{x}$ quantifies the fuzziness induced by the deficiency of information and passes it over to the interpolating function and hence increases the certainty about the interpolated value. This exploits all the used information content explicitly, which is otherwise represented only implicitly in the interpolating function. This procedure must be slightly modified in accordance with the applied interpolation method. However, the basic idea suggested here is generally valid.The one dimensional linear interpolation is again considered in the next sections as an introductory example to simplify matters and to demonstrate the way of finding the fuzzy function that fulfills the third requirement.

### 4.2.2 Case 1: Interpolating data with crisp quantities at crisp locations

The fuzzy interpolating function of the form $\widetilde{\widetilde{f}} : \mathbb{R} \rightarrow \widetilde{\mathbb{R}}$, given as

$$\widetilde{\widetilde{f}}(x) \ = \sum_{i=1}^{n} y_i \cdot \widetilde{\phi}_i(x) \tag{35}$$

with the fuzzy basis functions of the form $\widetilde{\phi}_i : \mathbb{R} \rightarrow \widetilde{\mathbb{R}}$, given as

$$\widetilde{\phi}_i(x) = \widetilde{\phi}_i(\widetilde{x}(x)) \tag{36}$$

can be used to interpolate data with crisp quantities $y_i$ at crisp locations $x_i$ with $i = 1, ..., n$ . The fuzzy basis functions $\widetilde{\phi}_i(x)$ depend on a fuzzy function $\widetilde{x}(x)$, given in Eq. 27, that maps every x to a fuzzy number. This fuzzy number quantifies the fuzziness and reduces the uncertainty.

The fuzzy linear interpolation of the data points $(x_1, y_1)$ and $(x_2, y_2)$, see Figure..., can be done using the fuzzy basis functions $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ given in Eq. 37 and Eq. 38, respectively.

$$\widetilde{\phi}_1(x) \ := \ \frac{x_2 - \widetilde{x}(x)}{x_2 - x_1} \tag{37}$$

$$\widetilde{\phi}_2(x) \ := \ \frac{\widetilde{x}(x) - x_1}{x_2 - x_1} \tag{38}$$

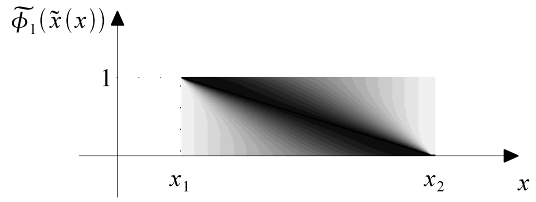


Figure 12: Basis function $\widetilde{\phi}_1$ .

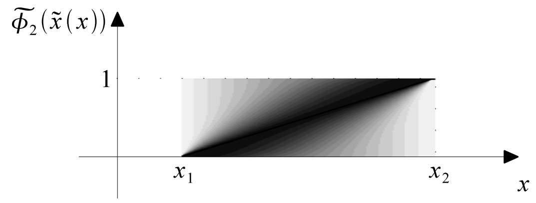


Figure 13: Basis function $\widetilde{\phi}_2$ .

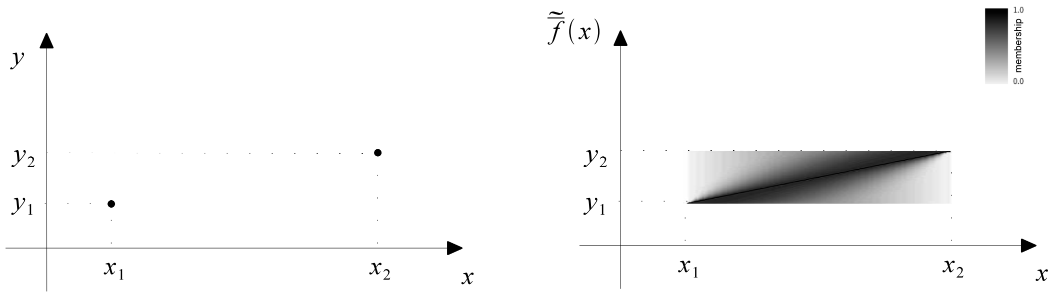


Figure 14: Fuzzy linear interpolation of crisp data.

The function $\widetilde{x}(x)$ is given in Eq. 27 and the membership function of the resulted fuzzy number is given in Eq. 28 . The graphs of $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ are shown in Figure 12 and Figure 13, respectively. The graph of the resulted fuzzy interpolating function is showen in Figure 14 .

The Characteristic-Density Factor can be used to modify the resulting fuzzy number to consider the density of the used data set and the characteristics of the studied region.

### 4.2.3 Case 2: Interpolating data with fuzzy quantities at crisp locations

Introducing the fuzziness induced by the information deficiency while interpolating data with fuzzy quantities at crisp locations can be done using the fuzzy interpolating function $\widetilde{\widetilde{f}} : \mathbb{R} \to \widetilde{\widetilde{\mathbb{R}}}$ given as

$$\widetilde{\widetilde{f}}(x) = \sum_{i=1}^{n} \widetilde{f}_i(x_i) \cdot \widetilde{\phi}_i(\widetilde{x}(x)) \tag{39}$$

The fuzzy basis functions $\widetilde{\phi}_i : \mathbb{R} \to \widetilde{\mathbb{R}}$ are given as

$$\widetilde{\phi}_i(x) = \widetilde{\phi}_i(\widetilde{x}(x)) \tag{40}$$

These fuzzy basis functions $\widetilde{\phi}_i$ depend again on the fuzzy function $\widetilde{x}(x)$ given in Eq. 27.

The fuzzy linear interpolation of the data points $(x_1, \widetilde{y}_1)$ and $(x_2, \widetilde{y}_2)$, see Figure 15, can be done using the fuzzy basis functions

$$\widetilde{\phi}_1(x) := \frac{x_2 - \widetilde{x}(x)}{x_2 - x_1} \tag{41}$$

$$\widetilde{\phi}_2(x) := \frac{\widetilde{x}(x) - x_1}{x_2 - x_1} \tag{42}$$

For the graphs of these basis function see Figure 1 and Figure 2.

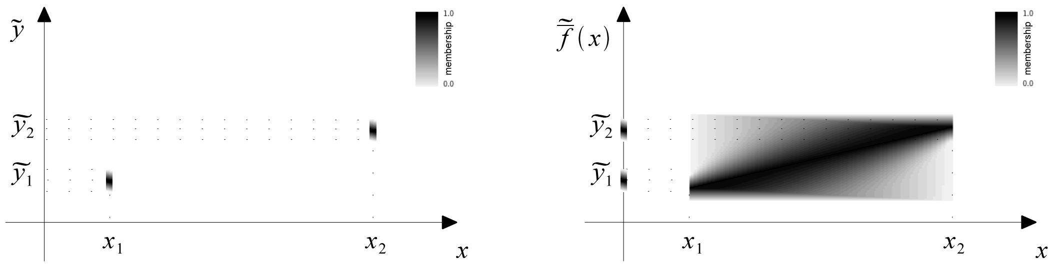


Figure 15: Fuzzy linear interpolation of fuzzy data.

The graph of the resulted fuzzy interpolating function is shown in Figure 15.

## 5 SHORT EXCURSION IN INTERPOLATION METHODS

In this section interpolation using different well-kown methods is considered. At first, sample points of the continuous analytical fuzzy function

$$\widetilde{f}(x) = \widetilde{a} \cdot \sin x \tag{43}$$

are interpolated by a Lagrange interpolant. After this, the interpolation of sample points of the same function in Eq. 43 is conducted by a piecewise linear interpolating function considering the fuzziness induced by the discrete and distributed state of the sample points. Original measured data are next interpolated by a piecewise bilinear interpolating function on a grid. The quantification of the fuzziness induced by the deficiency of information is regarded here, too. At last the Shepard interpolation method is used to interpolate sample points of the function in Eq. 43 again.

A graph of the function in Eq. 43 in the interval $[0, 2\pi]$ for the constant

$$\widetilde{a} = \{(\xi, \mu(\xi)) | \xi \in \mathbf{R}, \mu(\xi) \in \mathbf{R} \to [0, 1]\}, \tag{44}$$

where

$$\mu_{\widetilde{a}}(\xi) = \begin{cases} \frac{\xi - 0.7}{0.3} & \text{for } 0.7 \le \xi \le 1.0 \\ \frac{1.3 - \xi}{0.3} & \text{for } 1.0 \le \xi \le 1.3 \\ 0 & \text{otherwise,} \end{cases} \tag{45}$$

is shown in Figure 16.

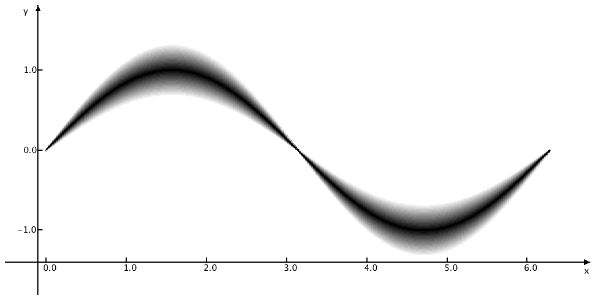


Figure 16: Fuzzy sine function.

### 5.1 Fuzzy Lagrange polynomial interpolation

Taking the constant $\widetilde{a}$ as given in Eq. 44 sample points of the function in Eq. 43 are taken at $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$ and $x = 2\pi$, see the left side of Figure 17.

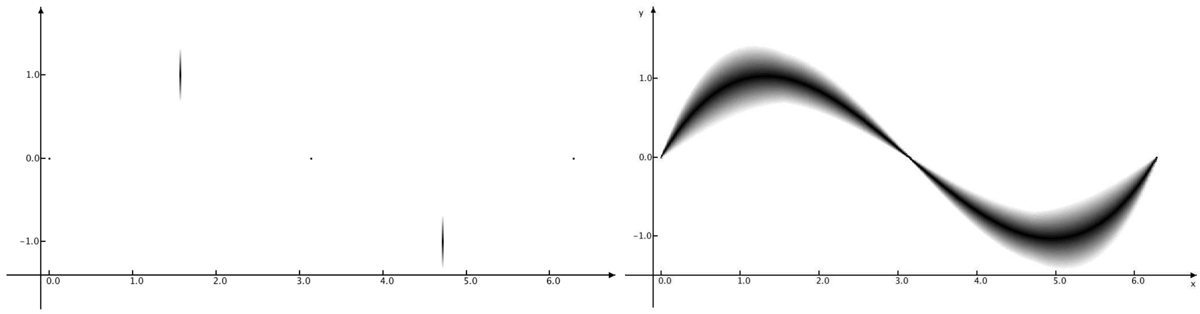

Figure 17: Fuzzy Lagrange polynomial interpolation of fuzzy sine function.

A fuzzy interpolating function of these sample points can be built using the Lagrange fundamental polynomials as basis function $\phi_i$ in

$$\widetilde{\widetilde{f}}(x) \ = \sum_{i=1}^{n} \widetilde{y}_i \cdot \phi_i(x) \tag{46}$$

The fundamental polynomials are given as

$$\phi_i(x) \ := \ \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \tag{47}$$

For the graph of the fuzzy Lagrange polynomial interpolating function see the right side of Figure 17.

### 5.2 Fuzzy piecewise linear interpolation

Here for the fuzzy piecewise linear interpolation two different instances of the continuous analytical fuzzy function in Eq. 43 are considered. The first instance is given for the single tone fuzzy number

$$\widetilde{a} = \{(\xi, \mu(\xi))|\xi \in \mathbf{R}, \mu(\xi) \in \mathbf{R} \rightarrow [0, 1]\}, \tag{48}$$

where

$$\mu_{\widetilde{a}}(\xi) = \begin{cases} 1 & \text{for } \xi = 1.0 \\ 0 & \text{otherwise,} \end{cases} \tag{49}$$

and hence $\widetilde{a}$ is a crip real number. This makes the fuzzy function a crisp real valued one. The graph of this crisp real function is shown in Figure 18.

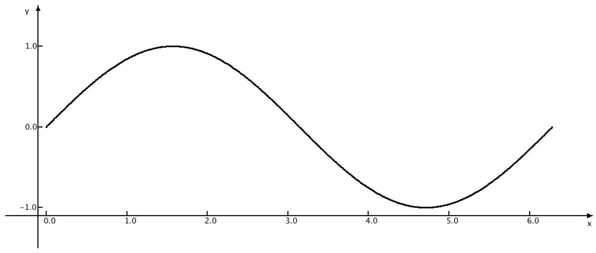


Figure 18: Crisp sine function.

Interpolating some sample points of this crisp real function using a classical piecewise linear interpolating function, see Figure 19, does not necessarily represent the original function accurately especially in the case of the total ignorance of the original function underlaying the sample points.

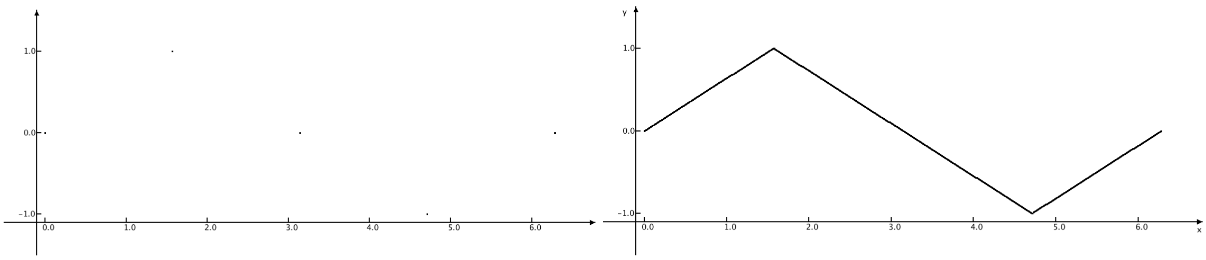


Figure 19: Crisp piecewise linear interpolation of a crisp sine function.

Here taking sample points at $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$ and $x = 2\pi$ again, see the left side of Figure 20, the fuzziness induced by the information deficiency is quantified and a fuzzy interpolating function for these sample points is constructed. The used basis functions $\widetilde{\phi}_i(\widetilde{x}(x))$ and $\widetilde{\phi}_{i+1}(\widetilde{x}(x))$ to build the interpolating function between two sample points $i$ and $i+1$

$$\widetilde{\widetilde{f}}(x) \;=\; y_i \cdot \widetilde{\phi}_i(\widetilde{x}(x)) + y_{i+1} \cdot \widetilde{\phi}_{i+1}(\widetilde{x}(x)) \tag{50}$$

are given as in the following.

$$\widetilde{\phi}_i(x) \;:=\; \frac{x_{i+1} - \widetilde{x}(x)}{x_{i+1} - x_i} \tag{51}$$

$$\widetilde{\phi}_{i+1}(x) \;:=\; \frac{\widetilde{x}(x) - x_i}{x_{i+1} - x_i} \tag{52}$$

The function $\widetilde{x}(x)$ is given in Eq. 27. For the graph of the interpolating function see Figure 20.

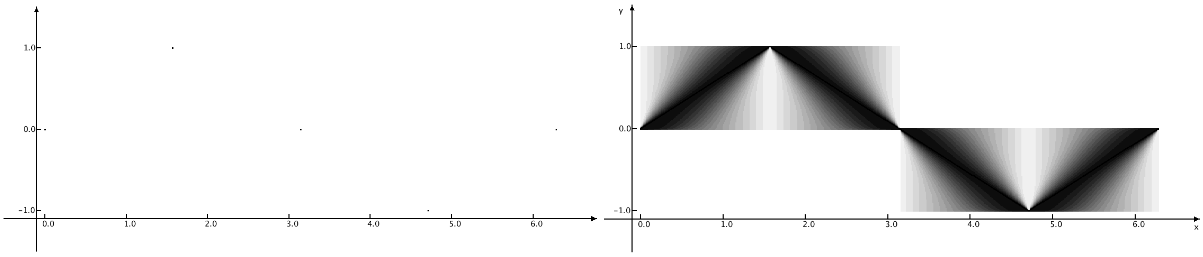

Figure 20: Fuzzy piecewise linear interpolation of a crisp sine function.

This interpolating function reduces the uncertainty in the resulted interpolation of the original function and reserves the whole information content. Refining the distance between the sample points increases the certainty in the interpolating function to a maximum level under the information available, see Figure 21 and Figure 22

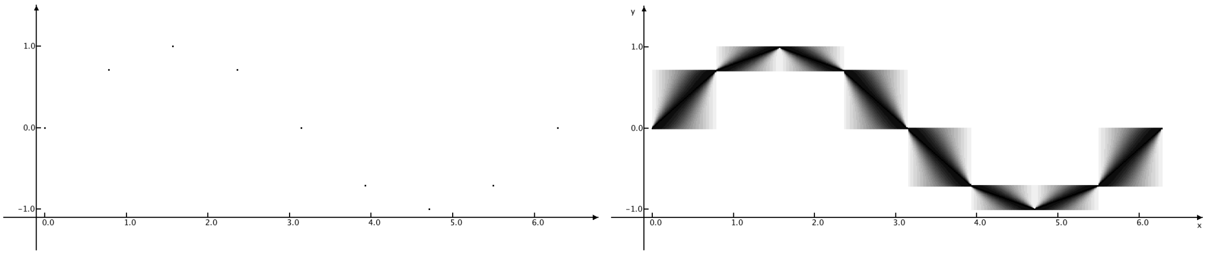

Figure 21: Fuzzy piecewise linear interpolation of a crisp sine function.

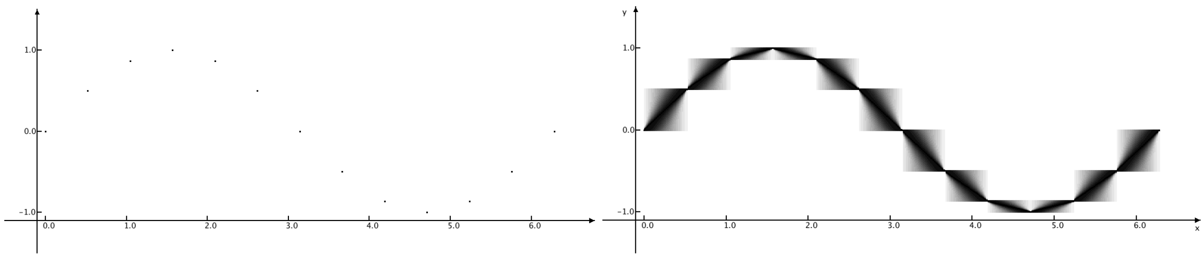

Figure 22: Fuzzy piecewise linear interpolation of a crisp sine function.

The second instance of the function in Eq. 43, which is considered here, is the same function as before resulted from $\widetilde{a}$ given by Eq. 44. A fuzzy piecewise linear interpolating function, which takes only the genuine fuzziness into account, is used to interpolate sample points at $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}i$ and $x = 2\pi$. The fuzzy interpolating function between two sampling points $i$ and $i+1$ is given as

$$\widetilde{\widetilde{f}}(x) \;=\; \widetilde{y}_i \cdot \phi_i(x) + \widetilde{y}_{i+1} \cdot \phi_{i+1}(x) \tag{53}$$

The basis functions $\phi_i(x)$ and $\phi_{i+1}(x)$ are given as in the following.

$$\phi_i(x) \;:=\; \frac{x_{i+1} - x}{x_{i+1} - x_i} \tag{54}$$

$$\phi_{i+1}(x) \;:=\; \frac{x - x_i}{x_{i+1} - x_i} \tag{55}$$

A graph of the interpolating function is shown in Figure 23

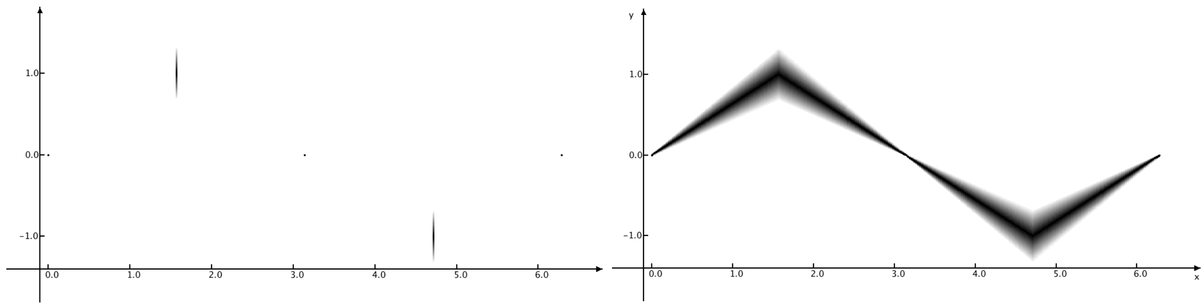


Figure 23: Fuzzy piecewise linear interpolation of a fuzzy sine function.

Quantifying the fuzziness induced by the deficiency of information and interpolating the same sample points using fuzzy interpolating function of the form

$$\widetilde{\widetilde{f}}(x) \;=\; \widetilde{y}_i \cdot \widetilde{\phi}_i(\widetilde{x}(x)) + \widetilde{y}_{i+1} \cdot \widetilde{\phi}_{i+1}(\widetilde{x}(x)) \tag{56}$$

reduces the uncertainty as before to a minimum with respect to the available information. The basis functions are given as in the following.

$$\widetilde{\phi}_i(x) \;:=\; \frac{x_{i+1} - \widetilde{x}(x)}{x_{i+1} - x_i} \tag{57}$$

$$\widetilde{\phi}_{i+1}(x) \;:=\; \frac{\widetilde{x}(x) - x_i}{x_{i+1} - x_i} \tag{58}$$

The function $\widetilde{x}(x)$ is given as before in Eq. 27. A graph of the interpolating function in Eq. 56

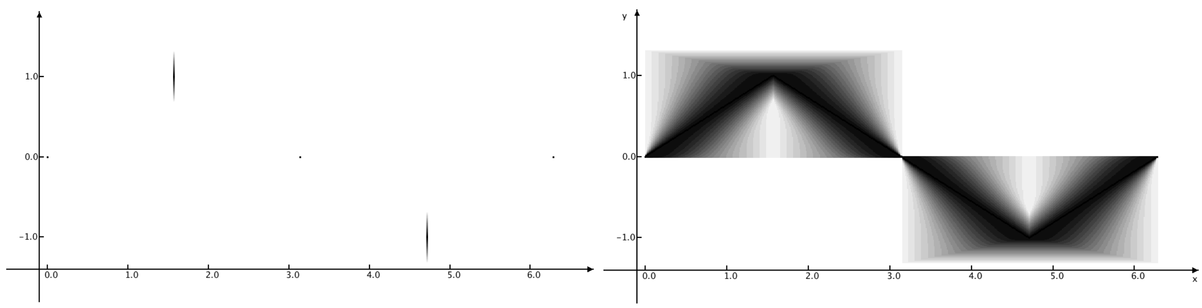


Figure 24: Fuzzy linear interpolating function.

is shown in Figure 24.

Again, with the refinement of the distance between the sampling points the interpolating function tends to be more certain and keeps the uncertainty at a minimum level, see Figure 25 and Figure 26.

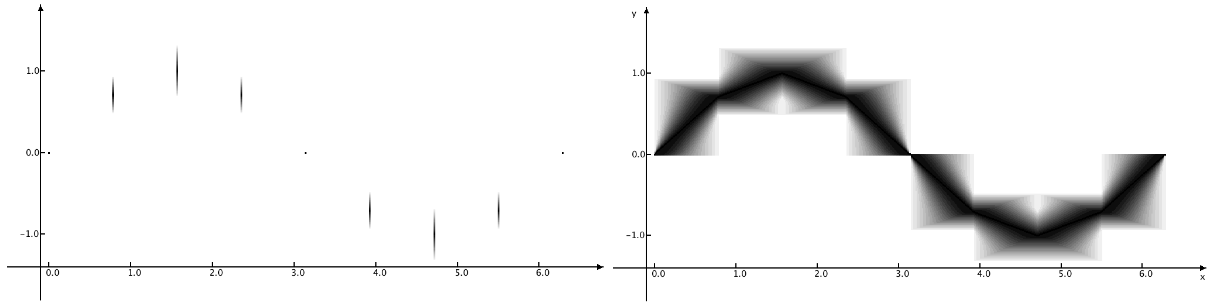


Figure 25: Fuzzy linear interpolating function.

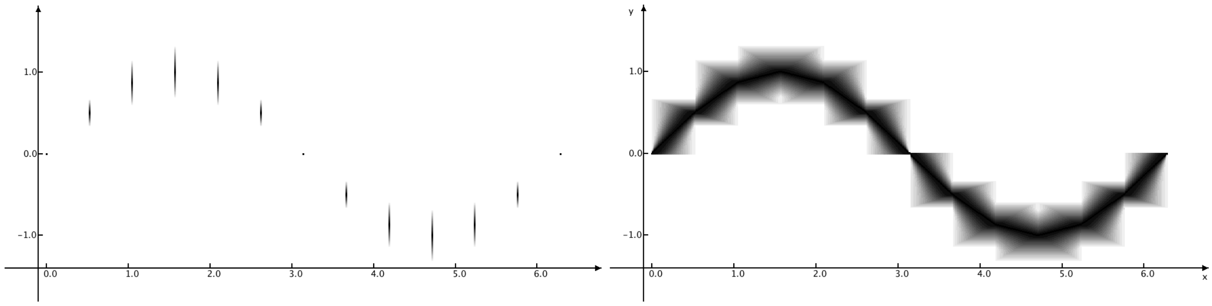


Figure 26: Fuzzy linear interpolating function.

### 5.3 Fuzzy piecewise bilinear interpolation on a grid

A two dimensional original bathymetric data set that covers the offshore area of the island of Langeoog off the German coast of the North Sea is interpolated here by a fuzzy piecewise bilinear interpolating function. The data set consists of a regular grid of 5 m distance and were resulting from fan echo sonar data over the year 2002. This data set is supplied by the Lower Saxony Water Management, Coastal Defence and Nature Conservation Agency (NLWKN) after treatment and processing which increases the uncertainty.

The fuzzy piecewise bilinear interpolating function between the points $(i, j)$, $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$ is given as in the following.

$$\widetilde{\widetilde{f}}(x,y) \;=\; \widetilde{z}_{ij}\cdot\widetilde{\phi}_{ij}(x,y)+\widetilde{z}_{i+1j}\cdot\widetilde{\phi}_{i+1j}(x,y)+\widetilde{z}_{ij+1}\cdot\widetilde{\phi}_{ij+1}(x,y)+\widetilde{z}_{i+1j+1}\cdot\widetilde{\phi}_{i+1j+1}(x,y) \quad (59)$$

The basis functions are given as in the following.

$$\widetilde{\phi}_{ij}(x,y) \;:=\; \frac{(y_{j+1}-\widetilde{y}(y))\cdot(x_{i+1}-\widetilde{x}(x))}{(x_{i+1}-x_i)\cdot(y_{j+1}-y_j)} \quad (60)$$

$$\widetilde{\phi}_{i+1j}(x,y) \;:=\; \frac{(\widetilde{y}(y)-y_{j+1})\cdot(\widetilde{x}(x)-x_i)}{(x_{i+1}-x_i)\cdot(y_{j+1}-y_j)} \quad (61)$$

$$\widetilde{\phi}_{ij+1}(x,y) \;:=\; \frac{(\widetilde{y}(y)-y_j)\cdot(x_{i+1}-\widetilde{x}(x))}{(x_{i+1}-x_i)\cdot(y_{j+1}-y_j)} \quad (62)$$

$$\widetilde{\phi}_{i+1j+1}(x,y) \;:=\; \frac{(y_{j+1}-\widetilde{y}(y))\cdot(\widetilde{x}(x)-x_i)}{(x_{i+1}-x_i)\cdot(y_{j+1}-y_j)} \quad (63)$$

The functions $\widetilde{x}(x)$ and $\widetilde{y}(y)$ are given as in Eq. 27. These basis functions and the resulting fuzzy bilinear interpolating function interpolate the genuine fuzziness in the data and quantify the fuzziness induced by the deficiency of information.

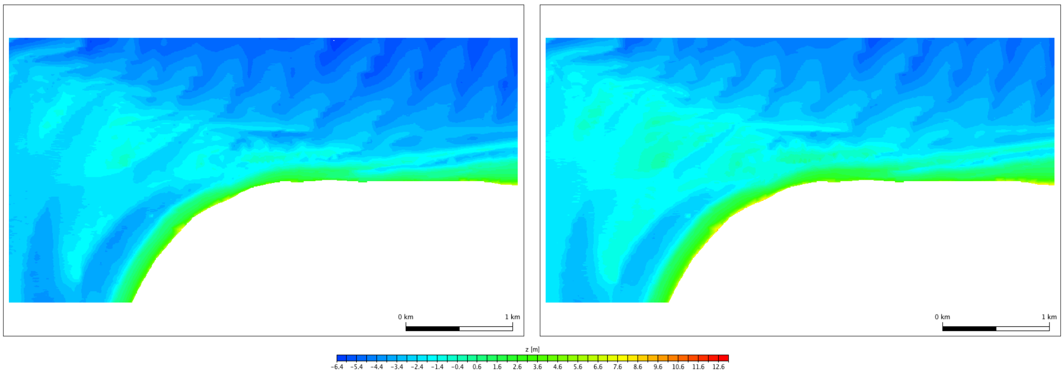


Figure 27: From left to right the minimum and maximum of the interpolated depth distribution in Spring 2002 at $0.0$ degree of membership.

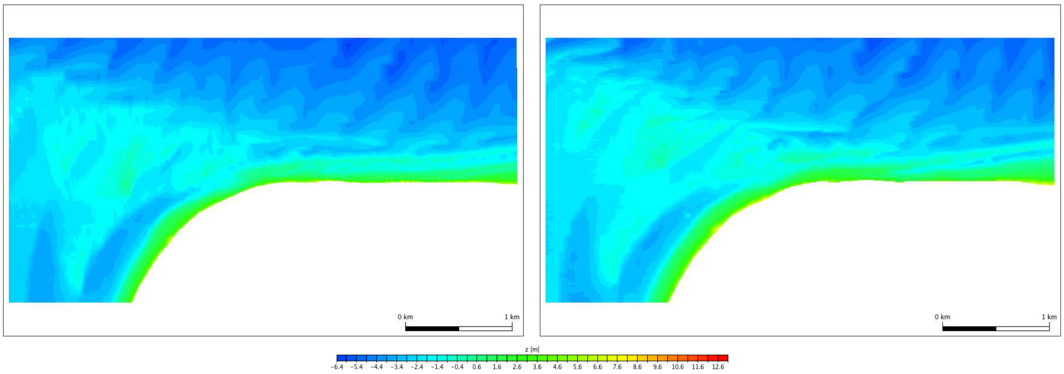


Figure 28: From left to right the minimum and maximum of the spatially interpolated depth distribution in Spring 2002 at $0.5$ degree of membership.

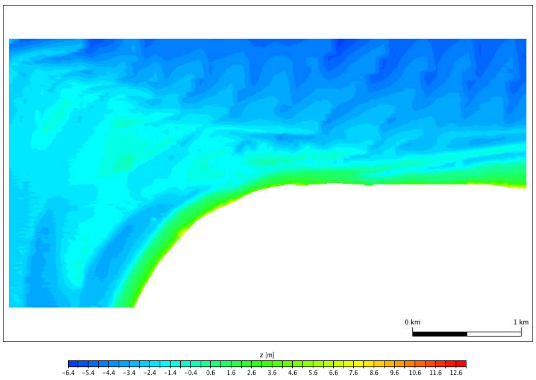


Figure 29: The spatially interpolated depth distribution in Spring 2002 at $1.0$ degree of membership.

Figures 27, 28 and 29 show the fuzzy interpolation of the depth in the Spring of the year 2002. In Figure 27 on the left side the minimum of the depth distribution at the $\alpha$-Cut level of $(0.0)$ is presented. The right picture shows the maximum of the depth distribution at the $\alpha$-Cut level of $(0.0)$. In Figure 28 on the left side the minimum of the depth distribution at the $\alpha$-Cut level of $(0.5)$ is presented. The right picture shows the maximum of the depth distribution at the $\alpha$-Cut level of $(0.5)$. Figure 29 shows the depth distribution at the $\alpha$-Cut level of $(1.0)$.

### 5.4 Fuzzy Shepard interpolation

At last if the basis functions $\phi_i$ with $i = 1, ..., n$ of a fuzzy interpolating function

$$\widetilde{\widetilde{f}}(x) \ = \sum_{i=1}^{n} \widetilde{y}_i \cdot \phi_i(x) \tag{64}$$

are defined depending on the distance $dist(x, x_i)$ between $x$ and $x_i$ and a smoothing parameter $\eta$, $0 < \eta < \infty$ as

$$\phi(x) \ := \ \frac{\varphi_i^{\eta}(x)}{\sum_{i=1}^{n} \varphi_i^{\eta}(x)}, \tag{65}$$

where $\varphi_i(x) = \frac{1}{dist(x,x_i)}$, the resulting interpolating function is a fuzzy Shepard interpolating function.

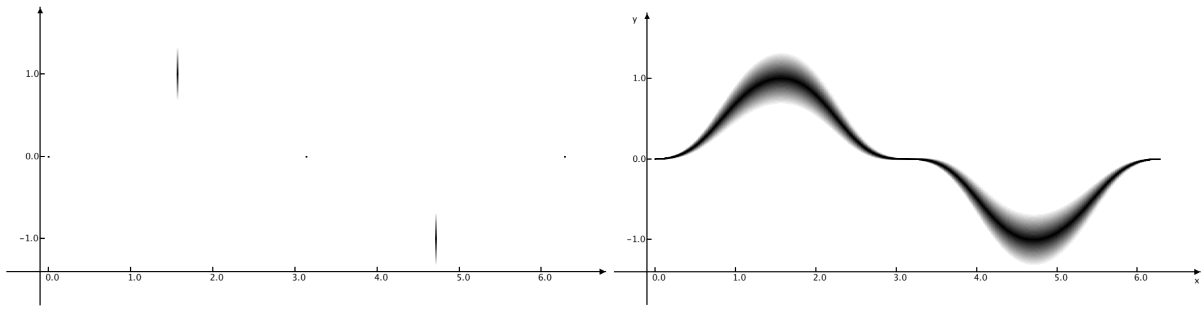


Figure 30: Shepard interpolation of a fuzzy fine function with $\eta = 2$.

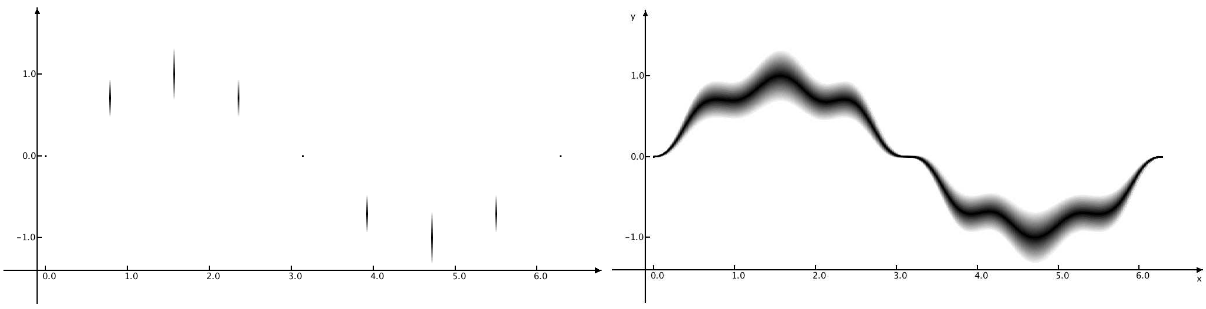


Figure 31: Shepard interpolation of a fuzzy fine function with $\eta = 2$.

If interpolating the sample points of the fuzzy function given in Eq. 43 taken at $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}i$ and $x = 2\pi$, see the left side of Figure 30, using the basis functions in Eq. 65 with $\eta = 2$, the function shown in Figure 30 results. Refining the distance between the sample points results in the interpolating function shown in Figure 31.

## 6 CONCLUSION

Uncertainties and fuzziness in scattered and discrete observations can be accounted for by fuzzy interpolating functions in a cohernt and closed form maintaining the informational content of the data set. In this paper the integration of the different uncertainties and fuzziness into a fuzzy valued function was discussed. Sets of artificial and original measured data were used to examine different fuzzy interpolation methods. The application of such fuzzy interpolation methods to interpolating bathymetric data might prove helpful in the case of the volumetric apporoach to calculating the sedimentation and erosion rates.

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